Closed-form Approximation to Comoving Distance vs Z

M.Lampton UCB SSL June 2005; Nov 2008, Mar & Sept 2010

It is well known [1] that in a flat universe, the radial and transverse comoving distances at redshift z are the same and are given by the expression

$$DC(z) = \int_0^z \frac{dy}{\sqrt{1+\Omega_m(3y+3y^2+y^3)}} \qquad (1)$$

where Ω_m is the mass fraction, currently believed to lie in the interval 0.25 < Ω_m < 0.30.
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Comoving distance is the basis of the geometrical inverse square law, and of the other measures of cosmological distance, namely the angular diameter distance

DA(z) = DC(z)/(1+z) (2)

and the luminosity distance

DL(z) = (1+z)*DC(z). (3)

For many purposes one needs to evaluate DC(z) using a straightforward method that avoids integration. Examples are spreadsheet estimates of cosmological fluxes and angular sizes, useful in planning and comparing potential cosmology experiments. As it turns out, there is no convenient analytical expression for the integral yielding DC(z). However there are a variety of analytical expressions [2],[3] that closely approximate the DL(z) or DC(z) function, and here I present two such functions.

One closed-form approximation DCapprox( ) that I recommend is

$$DCapprox(z,\Omega_m) = \frac{z}{\sqrt{1+az+bz^2}}$$
$$\text{where} \quad a = AA*\Omega_m \qquad (4)$$
$$\text{and} \quad b = \mathrm{BB}*\sqrt{\Omega_m}$$

For 0<z<2.5 and 0.26< Ω_m <0.28, the coefficients AA and BB that best fit the integral are

$$\begin{aligned} AA &= 1.718 \quad \text{and so for } \Omega_m = 0.27, \quad a = 0.464 \\ BB &= 0.315 \quad \text{and so for } \Omega_m = 0.27, \quad b = 0.164 \end{aligned} \qquad (5)$$

For these choices of coefficients, the RMS error is 0.002 when 0<z<2.5.

In Fig 1 below, I show a plot of DL(z), DC(z), and DA(z) top to bottom in dimensionless units. In blue, I plot the exact integrals DC(z) etc; in red I plot the DCapprox(z) variants, for choices of Ω_m =0.20, 0.25, 0.30, and 0.35 moving downward within each group.

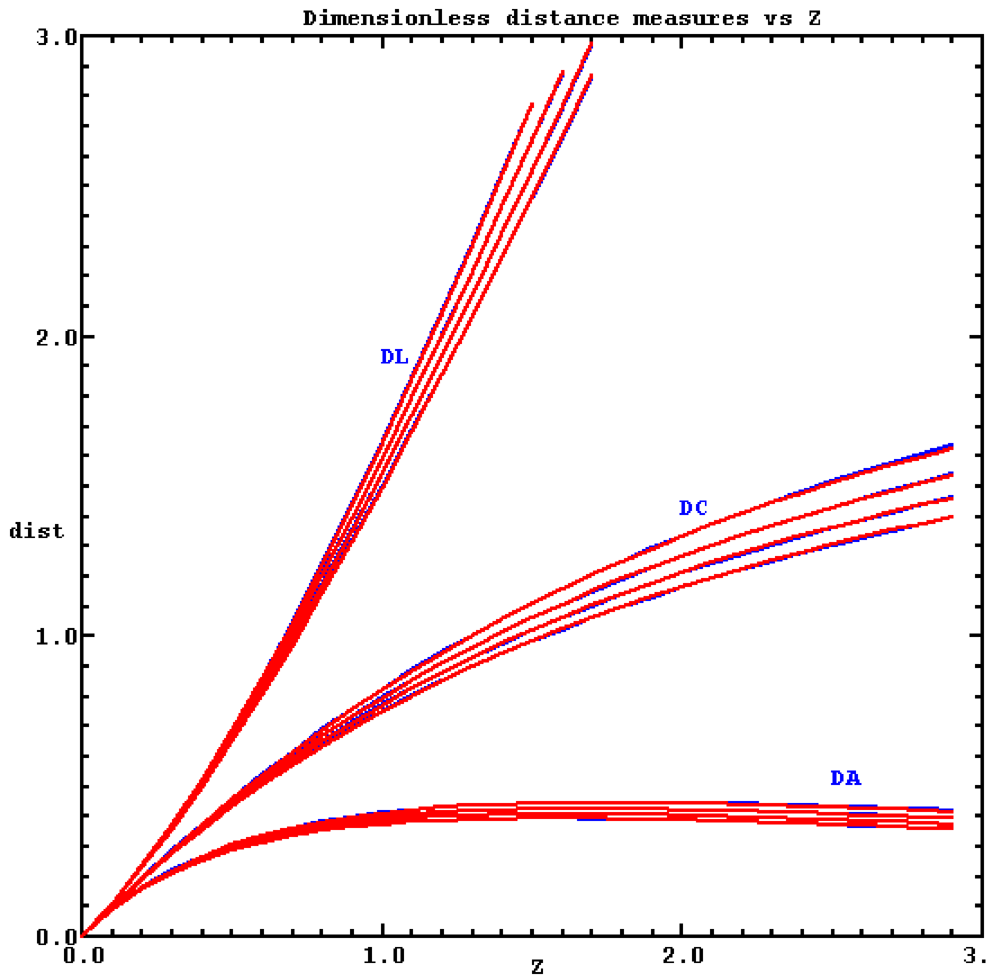

Fig 1. Top group: DL(z) for Ω_m =0.20, 0.25, 0.30, and 0.35. Middle group: DC(z) for the same Ω_m. Bottom group: DA(z) for the same Ω_m. Within each group the blue curves are the precise integrals (hard to see! sorry) and the red curves overlying them are the approximate forms based on my DCapprox(z). The RMS error in the approximation is 0.002 for this range of parameters.

In other situations you may need a closed form approximation to DC which is valid over the entire range of redshift, $0<z<\text{infinity}$. Since DC reaches an asymptotic value at large redshift that depends on Ω_m (for example it is 3.45 for Ω_m =0.27) it is important that the higher exponent of z within the approximation square root remain exactly 2.0. However the middle term in the square root can be adjusted both in coefficient and exponent for best fit over the complete range in redshift. When this is done for Ω_m =0.27, the resulting approximation becomes

$$DCapprox(z,\Omega_m = 0.27) = \frac{z}{\sqrt{1 + 0.4704 \cdot z^{1.3179} + 0.0854 \cdot z^2}} \qquad (6)$$

Here the RMS error over a set of 80 logarithmically spaced z-values 0.001 to 1E5 is 1%.

For most purposes you will want to convert these dimensionless distances to linear units. Multiply them by the Hubble distance D_H = 3Gpc/h = 4.29Gpc = $1.325\text{x}10^{26}$m when H_0 = 70km/s.Mpc and h = H_0/100. Once you have put the distance measures into linear units, you may apply the usual formulas: for example, power flux = Luminosity/$4\pi D_L^2$.

[1] D.Hogg, Astro-ph/9905116 (1999) and references therein.
[2] U. Pen, Astro-ph/9904172 (1999).
[3] T.Wickramasinghe & T.N.Ukwatta, arXiv:1003.0483 (2010).

A note added on Distance Modulus, ABmag, and STmag

Bolometric magnitudes include the radiant flux at all frequencies and wavelengths. The distance modulus (DM) is a logarithmic measure of the ratio of power flux of an object at cosmological redshift to its flux at a standard 10 pc distance. Thus for m_{bol} (observed) and M_{bol} (absolute, i.e. at 10 pc distance), both being measures of power flux,

$$m_{bol} = M_{bol} + DM \qquad (7)$$

where DM = distance modulus = $5 \log(D_L/10pc) = 5 \log(D_C/10pc) + 5 \log(1+z)$ where D_L and D_C have dimensions of length. Notice how the inverse square law appears in terms of DM: D_L^2 includes the inverse square law through D_C^2 (the geometric part) and also has two powers of (1+z), one for the time dilation and the other for the energy shift.
If, instead of power flux, we were interested in time-integrated energy in a pulse (say a gamma ray event), the distance relation would would contain the geometry and the energy shift but not the time dilation: $e_{bol}=E_{bol} + 5 \log(D_C/10pc) + 2.5 \log(1+z)$.
Or if we were interested in total steady photon flux over all wavelengths, the distance relation would contain the geometry and the time dilation but not the energy shift factor, so that $r_{bol} = R_{bol} + 5 \log(D_C/10pc) + 2.5 \log(1+z)$. And if, instead of photon rate, we were interested in the time-integrated number of photons in a gamma ray burst, the time dilation factor would vanish as well, resulting in an even simpler expression $n_{bol} = N_{bol} + 5 \log (D_C/10pc)$ having no (1+z) factors.

Anyway if we substitute the dimensionless D_C approximation from eqn (4), suitably multiplied by D_H=3/h Gpc, and take the logarithms, we get…

$$DM = 40 + 5 \log(3/h) + 5 \log[z/\sqrt{(1+az+bz^2)}] + 5 \log (1+z). \qquad (8)$$

When considering continuum fluxes per band or per frequency or wavelength interval, a K-correction must be introduced [1]:

$$m = M + DM + K. \qquad (9)$$

K is a magnitude error that would occur if the sizes and shapes of the emitting band (defining M) and the observing band (defining m) fail to obey $\lambda obs = (1+z)\cdot\lambda em$. If they were scaled appropriately, or were bolometric, with all wavelengths scaled by (1+z), then K=0. However, monochromatic fluxes (per Hz or per Angstrom) have no such scaling and their compression towards low frequencies (and dilution towards long wavelengths) appears explicitly as a system-dependent K-correction:

$$ABmag[\nu_{obs}] = ABmag[(1+z)\nu_{obs}, 10pc] + DM - 2.5 \log(1+z) \qquad (10)$$
$$STmag[\lambda_{obs}] = STmag[\lambda_{obs}/(1+z), 10pc] + DM + 2.5 \log(1+z). \qquad (11)$$

Evaluating the comoving distance numerically

Note added Sept 2010, M.Lampton

I am sometimes asked how to evaluate the comoving distance for a given cosmology. It is not widely appreciated that this potentially improper integral, shown above as equation (1), requires two changes of variable to make it generally tractable.

The first change of variable is well known to cosmologists. Define the scale factor "a"

$$a = 1/(1+z) \tag{12}$$

which maps $0<z<\infty$ onto $0<a<1$. This transformation moves the impropriety from the horizontal extension to infinity to a vertical extension at a=0. With $a^*=1/(1+z^*)$:

$$DC(a^*) = \int_{a^*}^{1} \frac{da}{\sqrt{\Omega_m a + (1-\Omega_m)a^4}} \tag{13}$$

Ordinary numerical integration tools (trapezoid, Simpson, Romberg) assume that the integrand and its derivatives are well behaved. They perform poorly when the integrand or its derivatives are infinite anywhere in the interval or at its endpoints.

The second change of variable is well known to numerical people (see Press et al, 4.4.5). The divergence of the integrand in equation (13) is a simple inverse square root, so define $a=t^2$ and rewrite the integral in terms of t:

$$DC(a^*) = 2\int_{\sqrt{a^*}}^{1} \frac{dt}{\sqrt{\Omega_m + (1-\Omega_m)t^6}} \tag{14}$$

In this form, the divergence of the integrand at the origin has been eliminated in favor of a nonlinear lower limit to the integral. The integrand is now well behaved even at a=0, and can be evaluated by ordinary numerical (trapezoid, Simpson, or Romberg) method.

Carrying out the first transformation, but not the second, is the reason that Ned Wright's very useful online Cosmology Calculator comes out a bit short when evaluating extreme distances, such as asking it the distance to the Big Bang.

For reference, I include a list of values of the dimensionless distance to the Big Bang for five values of the current mass fraction Ωm evaluated using (14) and a Simpson rule.

OmegaM=	0.100000:	DC(0)=	5.108744
OmegaM=	0.200000:	DC(0)=	3.890726
OmegaM=	0.300000:	DC(0)=	3.305076
OmegaM=	0.400000:	DC(0)=	2.938520
OmegaM=	0.500000:	DC(0)=	2.679595

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