ACUSTICA
S. HIRZEL VERLAG · STUTTGART

Volume 39 | 1978 | No. 4

Transmission Matrices in Electroacoustics

by M. Lampton

Space Sciences Laboratory, University of California, Berkeley, California 94720, USA

Summary

Transmission matrices (also known as transformation matrices, T-matrices, and $ABCD$-matrices) are traditionally employed in electrical circuit theory and mechanical vibration theory to mathematically evaluate the properties of cascaded-element networks of arbitrary complexity. The fundamentals of T-matrix representations are briefly reviewed, and extended to include transducers, lumped- and distributed-constant acoustical elements, and radiation. This extension allows complete electro-acoustic systems to be analyzed in a systematic way. Several examples of T-matrix representations are given, including microphones, loudspeakers, and coupled transducers. A further generalization allows branched networks to be similarly analyzed. Applications of the technique to the analysis, synthesis, and computer evaluation of electroacoustic systems are discussed.

Übertragungsmatrizen in der Elektroakustik

Zusammenfassung

Übertragungsmatrizen (auch bekannt als Transformationsmatrizen, T-Matrizen, $ABCD$-Matrizen) werden traditionell in der elektrischen Netzwerktheorie und in der Theorie mechanischer Schwingungen zur mathematischen Auswertung der Eigenschaften von Netzwerken mit kaskadierten Elementen und beliebiger Komplexität verwendet. Die Grundlagen der T-Matrix-Darstellung werden kurz wiederholt und so erweitert, daß sie Wandler, akustische Bauelemente im quasistationären und nichtstationären Betrieb und Abstrahlung einschließen. Diese Erweiterung erlaubt die systematische Analyse vollständiger, akustischer Systeme. Es werden einige Beispiele für T-Matrix-Darstellungen gegeben, die Mikrophone, Lautsprecher und gekoppelte Wandler enthalten. Eine weitere Verallgemeinerung gestattet in ähnlicher Weise die Analyse verzweigter Netzwerke. Es werden Anwendungen dieser Technik auf die Analyse, Synthese und Computerauswertung elektroakustischer Systeme diskutiert.

Les matrices de transmission en électroacoustique

Sommaire

Il est classique dans la théorie des circuits électriques et dans celle des vibrations mécaniques d'avoir recours aux matrices de transmission, dites également matrices de transformation, ou matrices T, ou matrices $ABCD$, pour calculer les propriétés de réseaux d'éléments en cascade aussi complexes qu'on le voudra. On passe brièvement en revue, dans le présent article, les principaux caractères des matrices T et on en étend l'usage pour représenter des traducteurs, des éléments acoustiques à constantes ponctuelles ou réparties, et des rayonnements. Cette généralisation permet de discuter de façon systématique des systèmes électroacoustiques complets. On donne plusieurs exemples de représentations au moyen de matrices T de microphones, de haut-parleurs et de traducteurs couplés. Une deuxième généralisation permet de traîter de même des réseaux maillés. On discute de l'application de cette technique à l'analyse, à la synthèse, et au calcul par ordinateurs de systèmes électroacoustiques.

1. Introduction

The analysis of linear systems is a frequently encountered chore in electrical, mechanical, and acoustical engineering. The usual method for obtaining a solution for the end-to-end response of a linear system is to represent the system as a network of circuit elements, and to write and solve a set of simultaneous equations governing all the variables in the network. Such a method is quite general, but can be rather complicated for large networks because one must deal with as many equations as there are variables in the system, in spite of the fact that one usually requires only the relationship between the input variables and the output variables. Moreover, if some modification is made to the network, the entire system of equations must again be solved. A considerably simpler procedure is, however, available, which applies to systems comprising cascaded linear stages. The method involves representing each stage or element of the system by its input-output characteristic, in the form of a transmission matrix (T-matrix). This representation is defined in such a way that the matrix of the entire cascaded system is the product of the T-matrices of its component stages. In this manner, the overall end-to-end properties of the system may be calculated by straightforward multiplication, without any need to solve coupled simultaneous equations and without reference to the intermediate internal variables.

T-matrix methods have enjoyed considerable use in electrical engineering in the analysis of lumped-element ladder circuits, power transmission systems, and waveguide problems [1], [2], [3]. They are also employed in mechanical engineering for the analysis of vibration [4], [5]. In this paper, I shall review the essential definitions, conventions, and features of T-matrix representations and show how transducers, distributed elements, and mechanical/acoustical interfaces are incorporated into the formalism. As an example, the response of a loudspeaker is derived from the matrix representations of its component parts. Several other topics, such as radiation coupling and nonladder systems, are also dealt with.

2. Electrical T-matrices

A linear two-port (four terminal) electrical network establishes two equations relating the voltage and current variables at its input (e_1, i_1) and its output (e_2, i_2). Under rather general conditions (see Appendix) these two equations may be written

$$e_1 = A\,e_2 + B\,i_2\,,$$
$$i_1 = C\,e_2 + D\,i_2 \tag{2-1}$$

or, in vector form,

$$\begin{pmatrix} e_1 \\ i_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} e_2 \\ i_2 \end{pmatrix}. \tag{2-2}$$

The four elements A, B, C, and D of the T-matrix completely characterize the network. $\boldsymbol{T}$ does not depend on the source or load impedances presented to it. The element A is the reciprocal of the open-circuit voltage gain of the network; it is dimensionless. B is the reciprocal of the network's short circuit transadmittance, and so has the dimensions of an impedance. C is the reciprocal of the open-circuit transimpedance, with units of admittance. Finally, D is the reciprocal of the short-circuit current gain, and is dimensionless.

The great utility of the T-matrix representation is the ease with which the end-to-end properties of cascaded systems may be evaluated. When two networks $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ are cascaded, the output of the first is the input of the second. Thus the T-matrix of the pair is simply the product of $\boldsymbol{T}_1$ with $\boldsymbol{T}_2$:

$$\boldsymbol{T} = \boldsymbol{T}_1\,\boldsymbol{T}_2 = \tag{2-3}$$
$$= \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}.$$

Successive multiplications give the input-output transformation for cascaded systems having arbitrarily many stages. One need not construct or analyze a complete schematic diagram of the system. A block diagram, or a list of the cascaded stages, suffices to establish the proper sequence in which the consecutive stage T-matrices are to be multiplied. Because matrix multiplication is not commutative, one must ensure that the matrix sequence does, in fact, correspond to the sequence of stages.

Two useful operations on T-matrices are inversion and reversion. The inverse, $\boldsymbol{T}^{-1}$, is defined as the matrix satisfying $\boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity matrix. In terms of $\boldsymbol{T}$,

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \dfrac{D}{\det T} & \dfrac{-B}{\det T} \\ \dfrac{-C}{\det T} & \dfrac{A}{\det T} \end{pmatrix} \tag{2-4}$$

where $\det T = AD - BC$ is the determinant of $\boldsymbol{T}$. The inverse exists whenever $\det T \neq 0$. Inverse matrices give output variables in terms of input variables, and are used to remove factors from an overall matrix product (that is, to accomplish division).

The reverse of a T-matrix represents a network whose input and output ports have been exchanged:

$$\boldsymbol{T}^{\mathrm{R}} = \begin{pmatrix} \dfrac{D}{\det T} & \dfrac{B}{\det T} \\ \dfrac{C}{\det T} & \dfrac{A}{\det T} \end{pmatrix}. \tag{2-5}$$

If a network is symmetrical with respect to its ports, $\boldsymbol{T}^{\mathrm{R}} = \boldsymbol{T}$; consequently if $B = C = 0$, then $A = \pm 1$ and $D = \pm 1$; otherwise, A must equal D and $\det T = 1$.

To find the input or output impedance of a system having known $\boldsymbol{T}$, one makes the load at the other port passive by setting $e = iZ_{\mathrm{L}}$; then (see [2] or [6])

$$Z_{\mathrm{in}} = \frac{AZ_{\mathrm{L}} + B}{CZ_{\mathrm{L}} + D} \quad \text{or} \quad Z_{\mathrm{out}} = \frac{DZ_{\mathrm{L}} + B}{CZ_{\mathrm{L}} + A} \tag{2-6}$$

from which one may recognize B/D as the short-circuit input impedance, A/C the open-circuit input impedance, B/A the short-circuit output impedance, and D/C the open-circuit output impedance.

Most linear passive networks exhibit reciprocal behaviour. By this is meant that if a source of current at one port causes some potential at a second port, then an equal current fed to the second port will cause an equal potential at the first port. This behaviour can be expressed in terms of the system's T-matrix. In the first case, the response $e_{\mathrm{out}}/i_{\mathrm{in}}$ is given by the expression

$$1/(AY_1 + BY_1Y_2 + C + DY_2)$$

Table I.
T-matrix representations of electrical two-port stages.

Expression	Name	T	det T	Z_{in}	Z_{out}	Remarks
(2—7)	Identity matrix	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$+1$	Z_L	Z_L	
(2—8)	Series impedance Z	$\begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix}$	$+1$	$Z+Z_L$	$Z+Z_L$	
(2—9)	Shunt admittance Y	$\begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix}$	$+1$	$(Y+Z_L^{-1})^{-1}$	$(Y+Z_L^{-1})^{-1}$	
(2—10)	Unity gain voltage follower	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	0	∞	0	active
(2—11)	Operational amplifier	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0	indet.	indet.	active
(2—12)	Positive immittance converter (transformer or PIC)	$\begin{pmatrix} n & 0 \\ 0 & 1/n \end{pmatrix}$	$+1$	$n^2 Z_L$	Z_L/n^2	passive if n real
(2—13)	Negative immitance converter (NIC)	$\begin{pmatrix} n & 0 \\ 0 & -1/n \end{pmatrix}$	-1	$-n^2 Z_L$	$-Z_L/n^2$	passive if n imaginary
(2—14)	Positive immittance inverter (gyrator or PII)	$\begin{pmatrix} 0 & Z \\ 1/Z & 0 \end{pmatrix}$	-1	Z^2/Z_L	Z^2/Z_L	passive if Z real
(2—15)	Negative immittance inverter (transverter or NII)	$\begin{pmatrix} 0 & Z \\ -1/Z & 0 \end{pmatrix}$	$+1$	$-Z^2/Z_L$	$-Z^2/Z_L$	passive if Z imaginary

where Y_1 and Y_2 are any arbitrary admittances which externally shunt ports 1 and 2. In the second case, the response is identical except for an overall factor of $\det T$. The ratio of the reverse to the forward response, $\det T$, is identified as the reciprocity index of the network. For reciprocal networks the responses are equal and $\det T = 1$. Because the determinant of a product of matrices equals the product of their determinants, a cascade of reciprocal networks is reciprocal.

Another class of linear systems exhibits anti-reciprocal behaviour [7], by which is meant that while the magnitudes of the forward and reverse transmissions are equal, their signs differ. In this case the reciprocity index $\det T = -1$. A unilateral system transmits in only one direction; for it, $\det T = 0$. An example is the ideal voltage follower (see Table I). In general, a linear system may have any value for $\det T$. However, electrical systems containing only linear passive elements are reciprocal if no static magnetic field is present.

Electrical networks are usually sufficiently complicated that all four of their matrix elements are non zero. However, individual elementary stages can often be identified whose T-matrix representations are simple. In Table I, I have listed some of the more useful examples, along with expressions for their input and output impedances when the other port is loaded with an arbitrary passive load impedance Z_L. Numerous further examples appear in the literature, e.g. [6] and references therein. The series and shunt immittance stages are particularly useful in assembling passive ladder networks. The idealized voltage follower and operational amplifier exemplify active stages. (The operational amplifier is, of course, used in branched feedback topologies rather than cascades. Its T-matrix is used with the branch expressions of section 7 below.)

The last four entries in Table I play a fundamental role in describing transducers. The immittance converters are parameterized by a dimensionless turns ratio relating port voltages and currents, and the impedance seen at either port is proportional to the impedance loading the other port. The simple passivity criterion[1] that if $\mathrm{Re}(Z_L) > 0$ then $\mathrm{Re}(Z_{in}) > 0$ requires that passive transformers have n real and passive NIC's have n imaginary. The two immittance inverters are parameterized by an impedance Z. For the PII (gyrator) to be passive requires that Z be real. For the NII (Bauer's transverter, [12]) to be similarly passive, its Z must be imaginary.

3. Mechanical and acoustical elements

The preceding formalism constitutes a systematic method for finding the end-to-end transfer characteristic of cascaded electrical networks, in terms of the characteristics of the component elements. The formalism may be extended to include transducers,

[1] Comprehensive discussions of network stability have been given by many authors. See [8] and references therein, [9], [10], and more recently [11].

mechanical components, acoustical devices, etc., by appropriately generalizing the column vectors so that they describe the variables at each stage of the system, and then defining each T-matrix in such a way as to correctly relate the required column vectors. The scheme can be made self-consistent provided that at each point the product of the two variables gives the rightward-directed power, in analogy with the electrical case. Furthermore, manipulation of expressions containing these various vectors is considerably simplified if they are kept in a unified impedance form, in which the upper component is the scalar drop variable and the lower one is the vector flow variable. With this rationale, the appropriate electrical variables are voltage and rightward flowing current, the mechanical variables are compressive force and rightward directed velocity, and the acoustic variables are pressure and rightward-directed volume velocity. With these choices, i.e. with

$$\begin{pmatrix} e \\ i \end{pmatrix} \text{ or } \begin{pmatrix} f \\ v \end{pmatrix} \text{ or } \begin{pmatrix} p \\ u \end{pmatrix},$$

the effect of a network element which supplies additional impedance is always described by a T-matrix for a series impedance, just as in the electrical case, expression (2-8). Similarly, the effect of a circuit element which admits additional flow is described by the basic admittance T-matrix, expression (2-9).

Elements of a mechanical circuit may be assembled into a T-matrix buildup by writing the mechanical mesh equations for forces and linear rightward-directed velocities in the form

$$\begin{pmatrix} f_1 \\ v_1 \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} f_2 \\ v_2 \end{pmatrix}. \tag{3-1}$$

In this form, a rigid connecting element possessing mass M, suspension stiffness S, and motional resistance R, will add to the system an impedance $Z = \mathrm{j}\,\omega M + R + S/\mathrm{j}\,\omega$. Its T-matrix representation is simply

$$\boldsymbol{T} = \begin{pmatrix} 1 & \mathrm{j}\,\omega M + R + S/\mathrm{j}\,\omega \\ 0 & 1 \end{pmatrix}. \tag{3-2}$$

Similarly, a nonrigid mechanical connecting link comprising an admittance Y has a T-matrix expressed by eq. (2-9). A lever obeys the transformer law given above in expression (2-12). In this manner, mechanical circuit elements can be mathematically cascaded by matrix multiplication as presented in section 2.

Connections between mechanical and acoustic circuits are made by pistons. An idealized piston can be regarded as a transducer between these regimes, obeying the relations connecting force, pressure, velocity, and volume velocity: $f = Sp$ and $v = u/S$. Here, S is the area of the piston. In T-matrix form these are written

$$\begin{pmatrix} f \\ v \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & 1/S \end{pmatrix} \begin{pmatrix} p \\ u \end{pmatrix} \tag{3-3}$$

or

$$\begin{pmatrix} p \\ u \end{pmatrix} = \begin{pmatrix} 1/S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} f \\ v \end{pmatrix}. \tag{3-4}$$

Such a device is clearly lossless, frequency independent, and reciprocal. Its T-matrix resembles that of an ideal transformer, with the turns ratio manifesting itself as a dimensional conversion [13]. If the piston is to be attributed a mechanical mass M say, one must multiply the T-matrix for M onto the "mechanical side" of the piston's T-matrix, e.g.:

$$\begin{pmatrix} 1 & \mathrm{j}\,\omega M \\ 0 & 1 \end{pmatrix} \begin{pmatrix} S & 0 \\ 0 & 1/S \end{pmatrix} = \begin{pmatrix} S & \mathrm{j}\,\omega M/S \\ 0 & 1/S \end{pmatrix}. \tag{3-5}$$

To incorporate acoustic circuit elements into the T-matrix method, one again writes the acoustical mesh equations for the input and output pressures and volume velocities in the form

$$\begin{pmatrix} p_1 \\ u_1 \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} p_2 \\ u_2 \end{pmatrix} \tag{3-6}$$

as was done for the mechanical elements. The impedance and admittance forms for T-matrices result from the acoustical mesh equations for impedances and admittances. The lumped-constant acoustical mass which characterizes a short duct of length L and area S with a fluid density ϱ_0 is

$$\begin{pmatrix} 1 & \mathrm{j}\,\omega\,\varrho_0 L/S \\ 0 & 1 \end{pmatrix} \tag{3-7}$$

as can be derived by multiplying a matrix for a mechanical mass $\varrho_0 LS$ on the left by matrix (3-4) and on the right by matrix (3-3). Similarly, if a small chamber of volume V is connected so as to shunt an acoustic flow, the chamber's additional admittance is described by the T-matrix

$$\begin{pmatrix} 1 & 0 \\ \mathrm{j}\,\omega\,V/\gamma\,P_0 & 1 \end{pmatrix} \tag{3-8}$$

where γ is the adiabatic index of the gas (1.4 for air) and P_0 is its static pressure. If instead the chamber encloses a piston, the added acoustic impedance is described by the matrix

$$\begin{pmatrix} 1 & \gamma\,P_0/\mathrm{j}\,\omega\,V \\ 0 & 1 \end{pmatrix}. \tag{3-9}$$

4. Transducers

We shall begin by considering a particularly important electromechanical transducer: the electrodynamic moving-coil motor. This device is characterized by a static magnetic field B and an electrical conductor of length l, whose product Bl sets the overall electromechanical coupling constant of the device. The transducer equations (see, for example, [8] or [13]) may be expressed as $e = Blv$ and $i = f/Bl$. In matrix form, these are

$$\begin{pmatrix} e \\ i \end{pmatrix} = \begin{pmatrix} 0 & Bl \\ 1/Bl & 0 \end{pmatrix} \begin{pmatrix} f \\ v \end{pmatrix}. \tag{4-1}$$

An idealized device of this form is lossless, reactanceless, and frequency independent. As is well known, the electrodynamic motor is an impedance inverter. This is manifested here by the skew form of the T-matrix. Electrodynamic motors are antireciprocal, as was first shown by McMillan [7]; that fact is manifested here by the determinant of the T-matrix being -1. The form of the T-matrix is that of the gyrator (see [8]).

More realistic and detailed models for electrodynamic motor behaviour can be built up with the matrix multiplication procedure described in section 2. For example, a series electrical resistance R can be put into the electrical input port by left-multiplying by the appropriate series impedance T-matrix, eq. (2-8). Similarly, mechanical forces due to suspension stiffness, inertia, and so forth, may be modelled by right-multiplying the motor's matrix by one describing the mechanical impedance Z_m. Finally, if the motor drives a piston of area S, the overall electroacoustic T-matrix for the resulting loudspeaker is

$$\boldsymbol{T} = \begin{pmatrix} \frac{RS}{Bl} & \frac{RZ_m}{SBl} + \frac{Bl}{S} \\ \frac{S}{Bl} & \frac{Z_m}{SBl} \end{pmatrix}. \tag{4-2}$$

This matrix, when applied to the acoustical output variables p and u, gives the electrical port voltage and current. Its determinant is -1, a symptom of the underlying electrodynamic process.

The sensitivity of the electroacoustic transducer described by eq. (4-2) may be evaluated if its acoustic load is specified. One such specification might be the simple infinite baffle, for which the acoustic load Z_a is the radiation load on both sides of the piston (e.g. twice Olson's eq. (5.13), [14]). This load may be explicitly incorporated as a system element by multiplying eq. (4-2) by the appropriate T-matrix, or may be dealt with implicitly by redefining the mechanical piston impedance Z_m to include the acoustically generated term $S^2 Z_a$. Either way, the acoustic output port is moved down-stream of the load, so that the output port pressure on which the system acts is zero. The sensitivity of the transducer for a given voltage excitation is then the reciprocal of the "B" matrix element, which with our simple model is

$$\frac{u_2}{e_1} = \frac{1}{B} = \frac{SBl}{RZ_m + (Bl)^2} \tag{4-3}$$

and its sensitivity for current excitation is the reciprocal of the "D" element, expressed here by

$$\frac{u_2}{i_1} = \frac{1}{D} = \frac{SBl}{Z_m}. \tag{4-4}$$

If a sealed enclosure of volume V blocks one side of the transducer's piston, the acoustic load will contain an enclosure impedance term, Z_b, in addition to the radiation impedance of the unblocked side of the piston. The enclosure's T-matrix representation has been given previously (3-9). The resulting sealed-enclosure loudspeaker system T-matrix is the product of eq. (4-2) with eq. (3-9), or

$$\boldsymbol{T} = \begin{pmatrix} \frac{RS}{Bl} & \frac{R}{SBl}(Z_m + S^2 Z_b) + \frac{Bl}{S} \\ \frac{S}{Bl} & \frac{1}{SBl}(Z_m + S^2 Z_b) \end{pmatrix}. \tag{4-5}$$

The loudspeaker described by eq. (4-2) can, of course, be operated in reverse as a microphone. The T-matrix appropriate to that connection is most easily calculated from eq. (4-2) with the help of the reverse matrix expression, (2-5). This gives

$$\boldsymbol{T} = \begin{pmatrix} \frac{Z_m}{SBl} & \frac{RZ_m}{SBl} + \frac{Bl}{S} \\ \frac{S}{Bl} & \frac{RS}{Bl} \end{pmatrix}. \tag{4-6}$$

This matrix acts on the output voltage and current to give the input pressure and volume velocity. An overall factor of -1 appears as a consequence of the antireciprocity of the basic motor. It has been removed by reversing the polarity of the electrical connection or the magnetic field. In any case, $\det T = -1$.

The sensitivity of this model microphone can be calculated once its electrical load is specified. Microphones are usually operated into a high electrical impedance load, for which $I_{out} = 0$. In this case, the pressure sensitivity, defined as the

ratio of the open circuit voltage to the incident pressure, is the reciprocal of the "A" element of the matrix; for our simple model,

$$\frac{1}{A} = \frac{SBl}{Z_m}. \tag{4-7}$$

Its acoustic input impedance is just $A/C = Z_m/S^2$.

The foregoing results are not new; the transducer equations have been known and understood for over a quarter century. A great deal of effort has, however, been expended towards the construction of equivalent electrical circuits for transducers (see, for example, [7], [8], [12], [13], [14], [15], and [16]). For instance, Hunt's ingenious $\varkappa$ operator [8] or β operator [13] provides a means of introducing antireciprocal behaviour into inherently reciprocal networks. The advantage of the T-matrix representation is that transducer characteristics may be obtained without the need for a detailed equivalent circuit, and, in the present context, without complications due to antireciprocal behaviour. An added bonus is that, for cascaded element systems, no sets of simultaneous equations need be solved; the product of the elements' T-matrices gives the T-matrix of the system, and hence the system's response, input impedance, and so forth.

Electrostatic transducers can also be described by T-matrices. These devices operate in such a manner that force is derived from an electrostatic signal charge q, and the terminal voltage is derived from the linear displacement x of the piston. The transducer's coupling constant is an electrostatic field, ε, in terms of which the transducer equations are $e = -\varepsilon x$ and $q = f/\varepsilon$. If we write $i = \mathrm{j}\,\omega q$ and $v = \mathrm{j}\,\omega x$, the desired input and output variables are obtained, and we have

$$\begin{pmatrix} e \\ i \end{pmatrix} = \begin{pmatrix} 0 & -\varepsilon/\mathrm{j}\,\omega \\ \mathrm{j}\,\omega/\varepsilon & 0 \end{pmatrix} \begin{pmatrix} f \\ v \end{pmatrix}. \tag{4-8}$$

In this idealized form, the transducer is lossless, frequency dependent, and reciprocal. Its skew form identifies it as an impedance inverter; indeed, its T-matrix is that of an NII. To include the series electrostatic capacitance of practical devices into the model, one must left-multiply the above T-matrix by the appropriate electrical series impedance form:

$$\begin{pmatrix} 1 & 1/\mathrm{j}\,\omega C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\varepsilon/\mathrm{j}\,\omega \\ \mathrm{j}\,\omega/\varepsilon & 0 \end{pmatrix} = \begin{pmatrix} 1/C\,\varepsilon & -\varepsilon/\mathrm{j}\,\omega \\ \mathrm{j}\,\omega/\varepsilon & 0 \end{pmatrix}. \tag{4-9}$$

A motor of this type is unstable. This can best be seen by use of the input impedance expression (2-6), which for this case gives

$$Z_{\mathrm{in}} = \frac{1}{\mathrm{j}\,\omega C} + \frac{\varepsilon^2}{\omega^2 Z_L} \tag{4-10}$$

where now Z_L is the mechanical impedance loading the diaphragm. If this load is an elastic spring of stiffness S, then $Z_L = S/\mathrm{j}\,\omega$ and

$$Z_{\mathrm{in}} = \frac{1}{\mathrm{j}\,\omega C} - \frac{\varepsilon^2}{\mathrm{j}\,\omega S} \tag{4-11}$$

which is the impedance of a negative (unstable) capacitor, if the stiffness is less than the critical stabilizing value $S_{\mathrm{crit}} = C/\varepsilon^2$ (see [8]).

The T-matrix for an electrostatic generator may be obtained from expression (4-9) by applying the reverse matrix transformation given in eq. (2-5). An electrostatic microphone may be modelled by adding a piston to the generator.

5. Distributed elements

The preceding expressions for the behaviour of various circuit elements are valid only in the limit that the dimensions of the element be substantially smaller than the wavelengths of the acoustical signals. Often, circuits contain elements which violate this criterion, with the consequence that distributed element (as opposed to lumped constant) representations must be employed.

Properties of distributed elements can be derived from the basic physical equations describing continuity of flow, conservation of momentum, and the gas compressibility law. In terms of the volume velocity u, the excess pressure p, and the excess density ϱ, these are

$$\text{continuity:} \quad \varrho_0 \frac{\mathrm{d}u}{\mathrm{d}x} + \mathrm{j}\,\omega S \varrho = 0 \tag{5-1}$$

$$\text{momentum:} \quad S \frac{\mathrm{d}p}{\mathrm{d}x} + (R + \mathrm{j}\,\omega \varrho_0) u = 0 \tag{5-2}$$

$$\text{compressibility:} \quad \varrho_0 p - \gamma P_0 \varrho = 0 \tag{5-3}$$

where linear interactions, proportional to $\exp(\mathrm{j}\,\omega t)$, have been assumed. These one-dimensional expressions are valid when the flow is parallel to the coordinate x. S is the cross-sectional area of the flow at each point. In eq. (5-2), the flow resistivity (if any) is represented by R.

Two methods for reducing these fluid equations to T-matrix form will be presented here. One method involves building up the T-matrix of a fluid in a duct by regarding it as a large number of thin

slabs in cascade; each slab has an identifiable T-matrix, and their product gives the matrix for the complete duct. Alternatively, the eigenfunction method involves finding two independent solutions for the above three differential equations; the variables at the input end of the duct can be found from these for any given values for the output variables.

The representation of a distributed one dimensional system in terms of a sequence of slabs is based on the T-matrix of a thin slab of arbitrary cross sectional area. If the thickness Δx of each slab is chosen to be substantially less than the wavelengths under consideration, then the phase shift across any slab will be small, and to first order in $\Delta x/\lambda$ its T-matrix will be

$$\begin{pmatrix} 1 & \Delta Z \\ \Delta Y & 1 \end{pmatrix} \tag{5-4}$$

where

$$\Delta Z = (\mathrm{j}\,\omega\,\varrho_0 + R)\,\Delta x/S$$

and

$$\Delta Y = \mathrm{j}\,\omega\,S\,\Delta x/\gamma\,P_0\,.$$

This matrix can be derived from eqs. (5-1) to (5-3) or from ordinary lumped-constant theory, as expressed by eqs. (3-7) and (3-8). The duct's T-matrix is the product of its component slab T-matrices,

$$\boldsymbol{T} = \prod \boldsymbol{T}_i = \boldsymbol{T}_1\,\boldsymbol{T}_2\,\boldsymbol{T}_3 \ldots \boldsymbol{T}_N\,. \tag{5-5}$$

This approach yields a general approximate numerical technique for evaluating the transmission of a duct or horn having a slowly varying but otherwise arbitrary shape, $S(x)$. One picks a slab thickness much smaller than the wavelength. Then, expression (5-5) is evaluated in which each $\boldsymbol{T}_i$ is (5-4) into which the appropriate cross sectional area has been substituted.

The product procedure becomes exact in the limit $\Delta x \to 0$, and can be used to find the exact analytic expression for $\boldsymbol{T}$ in a simple but important case: that of the uniform duct. This solution is made possible by the properties of two matrix functions (a comprehensive review of matrix functions is the text by Frazer et al. [17]). First, if $\boldsymbol{M}$ is a square matrix, define the matrix

$$\log \boldsymbol{M} = (\boldsymbol{M} - \boldsymbol{I}) - (\boldsymbol{M} - \boldsymbol{I})^2/2 + \\ + (\boldsymbol{M} - \boldsymbol{I})^3/3 - \cdots \tag{5-6}$$

in analogy with the Taylor's series for the ordinary logarithm. Here, $\boldsymbol{I}$ is the identity matrix, and the exponents of $(\boldsymbol{M} - \boldsymbol{I})$ signify self-multiplications. Whenever two matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ commute (i.e. whenever $\boldsymbol{MN} = \boldsymbol{NM}$) the matrix logarithm function has the same combinatorial property as the ordinary logarithm, viz. $\log \boldsymbol{MN} = \log \boldsymbol{M} + \log \boldsymbol{N}$. Second, define

$$\exp \boldsymbol{M} = \boldsymbol{I} + \boldsymbol{M} + \boldsymbol{M}^2/2! + \boldsymbol{M}^3/3! + \cdots \tag{5-7}$$

in analogy with the ordinary exponential function. This matrix series converges for all matrices $\boldsymbol{M}$. Its combinatorial property is, for matrices $\boldsymbol{M}$ and $\boldsymbol{N}$, $\exp(\boldsymbol{M} + \boldsymbol{N}) = \exp(\boldsymbol{M}) \exp(\boldsymbol{N})$ whenever $\boldsymbol{M}$ and $\boldsymbol{N}$ commute. Moreover, the matrix exponential undoes the matrix logarithm:

$$\exp(\log \boldsymbol{M}) = \log(\exp \boldsymbol{M}) = \boldsymbol{M}.$$

The exact solution for the uniform duct problem is obtained in the following way. By taking the matrix logarithm of both sides of eq. (5-5), one finds

$$\log \boldsymbol{T} = \sum_1^N \log \boldsymbol{T}_i = N \log \boldsymbol{T}_i\,. \tag{5-8}$$

As $\Delta x \to 0$, $\boldsymbol{T}_i$ approaches the identity matrix, and $\log \boldsymbol{T}_i$ can be trivially evaluated from expression (5-6). This evaluation gives

$$\log \boldsymbol{T} = N \begin{pmatrix} 0 & \Delta Z \\ \Delta Y & 0 \end{pmatrix} = \\ = \begin{pmatrix} 0 & N\Delta Z \\ N\Delta Y & 0 \end{pmatrix} = \begin{pmatrix} 0 & Z \\ Y & 0 \end{pmatrix}. \tag{5-9}$$

Here, the duct length is $L = N\Delta x$, the duct total impedance $Z = (\mathrm{j}\,\omega\,\varrho_0 + R)\,L/S$, and the duct admittance $Y = \mathrm{j}\,\omega\,LS/\gamma\,P_0$. Finally, the duct T-matrix is just the matrix exponential of expression (5-9). It is most conveniently evaluated by summing the infinite series (5-7) to give

$$\boldsymbol{T} = \begin{pmatrix} \cos KL & \mathrm{j}Z_0 \sin KL \\ \dfrac{j}{Z_0} \sin KL & \cos KL \end{pmatrix} \tag{5-10}$$

where the complex propagation number K is

$$K = \sqrt{\frac{\omega^2\,\varrho_0 - \mathrm{j}\,\omega\,R}{\gamma\,P_0}} \tag{5-11}$$

and where the characteristic impedance is

$$Z_0 = \frac{\varrho_0\,c}{S} \sqrt{1 + \frac{R}{\mathrm{j}\,\omega\,\varrho_0}}\,. \tag{5-12}$$

The lossy duct is clearly frequency dependent but reciprocal. The lossless case is treated by setting $R = 0$; then $K \to k \equiv \omega/c$, and expression (5-12) reduces to the familiar acoustic transmission line impedance $\varrho_0 c/S$. From expression (5-10), the $\tan(kL)$ behaviour of the short circuit duct impedance and the $\cot(kL)$ behaviour of the open circuit input impedance are easily derived.

The logarithm method for evaluating distributed systems is limited to uniform ducts. This limitation arises from the fact that a product of matrices, such as (5-5), can only be written in terms of a sum of matrices such as (5-8) when they commute. If the elementary slab matrices differed, their multiplication sequence would affect the result, yet the addition sequence of their logarithms would be immaterial. The analysis of horns and nonuniform ducts requires the use of more general methods.

The eigenfunction method provides a more powerful technique. The first step in this procedure is to reduce eqs. (5-1) to (5-3) to a single equation in one variable. In terms of (say) the excess pressure p, the resulting horn equation is

$$\frac{\mathrm{d}^2 p}{\mathrm{d}x^2} + \frac{\mathrm{d}p}{\mathrm{d}x}\frac{\mathrm{d}\log S}{\mathrm{d}x} + \frac{\omega^2 \varrho_0 - \mathrm{j}\,\omega R}{\gamma p_0} p = 0\,. \tag{5-13}$$

The second step is to identify two independent solutions (eigenfunctions) of this equation. Call these p_f and p_g. The general pressure field in the system is a combination of these solutions, for example $f p_f + g p_g$. For any such combination, the velocity field can be calculated using eq. (5-2). The third step in this procedure is to find the values of f and g which correspond to an arbitrary pair of variables at port 2. With these particular values substituted into $f p_f + g p_g$, the field everywhere in the device is obtained, and the input variables p_1 and u_1 are the field values at the location of the input port. Each is a function of p_2 and u_2. The elements of the T-matrix are these four dependences.

One application of the eigenfunction method is to obtain the transfer characteristics of horns. The exponential horn will be treated here, as an example of the general method. An exponential horn has a cross sectional area which grows exponentially with distance: $S(x) = S_0 \exp(mx)$, with m defining the flare rate of the horn. Thus $\mathrm{d}\log S/\mathrm{d}x = m$. The eigenfunctions of eq. (5-13) combine to give

$$p(x) = f \exp(-\mathrm{j}\,K - m/2)\,x + \\ + g \exp(\mathrm{j}\,K - m/2)\,x \tag{5-14}$$

with the complex propagation number K given by

$$K = \sqrt{\frac{\omega^2 \varrho_0}{\gamma P_0} - \frac{\mathrm{j}\,\omega R}{\gamma P_0} - \frac{m^2}{4}}\,. \tag{5-15}$$

With the help of eq. (5-2) the volume velocity $u(x)$ may be found. Then, the arbitrary constants f and g are fixed by the pressure and velocity at port 2, which for convenience may be taken to lie at $x = 0$. With this definition, S_0 corresponds to the mouth area of the horn, and if port 1 is to be the throat at $x = -L$, we obtain p_1 and u_1 from $p(-L)$ and $u(-L)$. The resulting T-matrix is (see [5]):

$$T = \begin{pmatrix} \mathrm{e}^{mL/2}\left(\cos KL - \dfrac{m}{2K}\sin KL\right) & \mathrm{e}^{mL/2}\dfrac{\mathrm{j}\,\omega \varrho_0 + R}{S_0 K}\sin KL \\ \mathrm{e}^{-mL/2}\dfrac{\mathrm{j}\,\omega S_0}{\gamma P_0 K}\sin KL & \mathrm{e}^{-mL/2}\left(\cos KL + \dfrac{m}{2K}\sin KL\right) \end{pmatrix}. \tag{5-16}$$

In this expression, the well-known impedance transformer behaviour is manifested in the factors of $\exp(mL/2)$ occurring in the A and B elements, and the factors of $\exp(-mL/2)$ in the C and D elements. Expression (2-6) shows that these lead to an overall impedance gain of $\exp(mL)$. The response irregularities of finite length horns are associated with the trigonometric factors in the matrix.

Another application of the horn equation is the conical horn, in which the wavefronts are sections of spherical surfaces subtending a solid angle Ω characteristic of the horn. The situation is of particular interest in the case of $\Omega = 4\pi$ for which the waves are complete spheres, appropriate for radiation into free space. In spherical problems such as this one it is customary to replace x by the variable r signifying radial distance from the origin. For this problem we shall also take the flow resistivity R to be zero, so that $K \to k = \omega/c$. With $S(r) = \Omega r^2$ the eigenfunctions of eq. (5-13) can be combined to give

$$p(r) = \frac{f}{r}\exp(-\mathrm{j}\,k r) + \\ + \frac{g}{r}\exp(+\mathrm{j}\,k r)\,. \tag{5-17}$$

The corresponding volume velocity is calculated in the usual way, and f and g are fixed by the given p and u at port 2, taken to lie at $r = b$. The variables at $r = a$ then establish the T-matrix for radiation between radii a and b:

$$T = \begin{pmatrix} \dfrac{b}{a}\cos\theta - \dfrac{1}{ka}\sin\theta & \dfrac{\mathrm{j}\,\varrho_0 c}{a b \Omega}\sin\theta \\ \dfrac{\mathrm{j}\,\Omega}{k^2 \varrho_0 c}\left[(1 + k^2 a b)\sin\theta - \theta\cos\theta\right] & \dfrac{a}{b}\cos\theta + \dfrac{1}{kb}\sin\theta \end{pmatrix} \tag{5-18}$$

where $\theta = kb - ka$ is the phase shift, in radians, between the throat and the mouth of the horn.

6. Free space radiation

A comprehensive treatment of an electroacoustic system will often require that acoustic propagation through a free space or bounded space region be considered as part of the system. For this reason, it is of interest to examine ways in which acoustic propagation might be represented by T-matrices.

The T-matrix method strictly applies only to two-port networks, which have clearly identifiable inputs and outputs. A free space medium cannot be described in such a simple way, because for each possible source location there exists an infinite number of output port locations. Moreover, although the concept of sound pressure is well defined in an infinite medium, the notion of total volume velocity requires careful treatment. The subject becomes yet more complex if bounded media are to be represented, for instance to characterize sound within an enclosed chamber. Accordingly, the present treatment will not consider representations of the continuous fluid itself, but rather will be restricted to a matrix description of the radiative coupling between transducers of various sorts.

Coupling of acoustic devices may be understood in terms of the behaviour of an isotropic point source. Such a device is a small spherical surface of radius $a \ll c/\omega$ which produces isotropic waves by oscillating radially at frequency ω. If the amplitude of the volume velocity of the source is u, the amplitude p of the pressure at distance d is proportional to u, with

$$\frac{p(d)}{u} = Z_r = \frac{j\,\omega\,\varrho_0}{4\pi d}\,e^{-jkd}\,. \tag{6-1}$$

This result is easily derived from the preceding material on spherical waves. It has been widely used in treatments of radiating systems, such as [13], p. 93. Consider now two such sources, separated a distance d in a common medium. If their self-impedances are Z_1 and Z_2, and they have volume velocities u_1 and u_2, then their pressures will be

$$p_1 = Z_1\,u_1 + Z_r\,u_2 \tag{6-2}$$

$$p_2 = Z_r\,u_1 + Z_2\,u_2\,. \tag{6-3}$$

These four volume velocity coefficients constitute an impedance matrix for the coupled sources. It may be used to find the acoustic T-matrix connecting the two points, using the relations given in the Appendix:

$$\boldsymbol{T} = \begin{pmatrix} Z_1/Z_r & Z_1 Z_2/Z_r - Z_r \\ 1/Z_r & Z_2/Z_r \end{pmatrix}. \tag{6-4}$$

In this matrix, it is the C element which gives the port-1 volume velocity required to produce unit pressure at distance d. The phase factor in $1/Z_r$ is $\exp(+jkd)$, which shows that the source volume velocity leads the port-2 pressure by a phase angle of kd radians. The A element gives the port-1 pressure associated with this flow, which is just CZ_1. Similarly, the D element gives the port-1 volume velocity required to give unit volume velocity at port 2; it is CZ_2. Finally, B contains two terms. The first of these is the port-1 pressure associated with unit volume velocity at port 2 due to radiation expanding from port 1; its phase factor shows p_1 leading u_2 by kd radians. The second term describes radiation from port 2 reaching port 1, with p_1 lagging u_2 by kd radians.

Expression (6-4) appears, at first sight, to be rather complicated. A substantial simplification is made possible by recognizing that it contains a characterization of each source in addition to the radiation field. Thus we may expect to be able to factor it into three consecutive parts: a matrix describing point source 1, a matrix describing the radiative coupling that connects the two locations, and a matrix describing point source 2. That is, we shall write expression (6-4) as

$$\boldsymbol{T} = \boldsymbol{T}_{Z1}\,\boldsymbol{T}_{\mathrm{rad}}\,\boldsymbol{T}_{Z2}\,. \tag{6-5}$$

To solve this expression for $\boldsymbol{T}_{\mathrm{rad}}$, we first left-multiply $\boldsymbol{T}$ by the inverse of the Z_1 series impedance T-matrix. Then we right-multiply that result by the inverse of the Z_2 series impedance T-matrix. The result of this factorization reveals

$$\boldsymbol{T}_{\mathrm{rad}} = \begin{pmatrix} 0 & -Z_r \\ 1/Z_r & 0 \end{pmatrix} \tag{6-6}$$

which is, as desired, independent of Z_1 and Z_2. Radiation coupling is clearly reciprocal; expression (6-6) has the form of the negative immittance inverter discussed in connection with Table I.

It is instructive to apply these isotropic radiation relationships to the calibration of small transducers by the free-field reciprocity procedure. This important calibration method has been widely discussed (see [14] or [18]). In this simplest form it can be used to find the absolute open-circuit pressure sensitivities of a pair of microphones, one of which is reciprocal (or antireciprocal) without initially knowing any absolute free-field pressure levels. Here, I shall analyze the procedure from the T-matrix viewpoint.

As was noted in section 4, the open circuit pressure sensitivity of a microphone is, quite generally, the reciprocal of its T-matrix "A" element. Thus, the calibration procedure may be viewed as a way of finding the A values of the

microphones being calibrated. This is accomplished in two steps. The first step is to subject both microphones to an equal (but unknown) sound pressure and note the ratio of their outputs; this fixes $A_2/A_1 = e_1/e_2$. The second step is to operate transducer 1 as a loudspeaker and measure the ratio of its drive current, i_{in}, to the open circuit output voltage of transducer 2 when they are separated in a free field by a distance d. The T-matrix of this three-stage system is

$$\boldsymbol{T} = \boldsymbol{T}_1^{\mathrm{R}}\, \boldsymbol{T}_{\mathrm{rad}}\, \boldsymbol{T}_2 \tag{6-7}$$

where $\boldsymbol{T}$ relates the system's electrical input and output through

$$\begin{pmatrix} e_{in} \\ i_{in} \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} e_{out} \\ i_{out} \end{pmatrix}. \tag{6-8}$$

The reverse matrix for the first transducer is generally given by expression (2-5), and T_{rad} is simply expression (6-6). With these substitutions,

$$\boldsymbol{T} = \frac{1}{\det T_1} \times \tag{6-9}$$
$$\times \begin{pmatrix} B_1 A_2/Z_r - D_1 C_2 Z_r & B_1 B_2/Z_r - D_1 D_2 Z_r \\ A_1 A_2/Z_r - C_1 C_2 Z_r & A_1 B_2/Z_r - C_1 D_2 Z_r \end{pmatrix}.$$

In this application, we wish to isolate the product $A_1 A_2$. This term appears in the third element of the matrix, where in the far field (Z_r small) it dominates the term containing $C_1 C_2$. Thus to find $A_1 A_2$ we measure the ratio

$$\frac{i_{in}}{e_{out}} = \frac{A_1 A_2 - C_1 C_2 Z_r^2}{Z_r \det T_1} \cong \frac{A_1 A_2}{Z_r \det T_1}. \tag{6-10}$$

The radiation transimpedance Z_r is directly calculable from eq. (6-1). If transducer 1 is reciprocal or antireciprocal, $\det T_1 = \pm 1$, and the procedure's second step yields the product $A_1 A_2$. Finally, the product is combined with the ratio from the first step to give

$$A_1^2 = \frac{e_2}{e_1} \frac{i_{in}}{e_{out}} Z_r \det T_1$$

and

$$A_2^2 = \frac{e_1}{e_2} \frac{i_{in}}{e_{out}} Z_r \det T_1. \tag{6-11}$$

Following this example, it becomes a straightforward matter to incorporate radiation coupling into the mathematical analysis of a variety of electroacoustic systems. Rather than pursue this topic further, I shall now turn to an important generalization of the T-matrix technique which allows a broader class of systems to be analyzed.

7. Branched systems

The T-matrix formulism provides a convenient way to evaluate cascaded systems, but a variety of important electroacoustic configurations contain branches or parallel paths. A circuit junction occurs wherever two paths separate or join. This junction is a simple three-port device, but three-port devices do not have T-matrix representations. Nonetheless, the mesh equations at these nodes are straightforward to solve, and allow a T-matrix to be written which exactly represents the combined effects of both branches. Such a combination T-matrix converts the branches into a single network stage.

Two kinds of three-port junctions are considered here. In a series junction, the voltages (or forces, or pressures) of two branches add to give the combined voltage of the junction. The currents in the branches are equal. In a parallel junction, the currents (or velocities, or volume velocities) of the branches add to give the combined current of the junction. The voltages in the branches are equal. Depending on the system being analyzed, either type of junction can occur at either end of the branch. Thus there are four situations to be analyzed.

Let the T-matrix of the first branch be denoted T and have elements A, B, C, and D. Let the second branch T' have elements A', B', C', and D'. If they are combined with parallel inputs and series outputs, the combination's mesh equations can be represented by

$$\boldsymbol{T}_{\mathrm{piso}} = \begin{pmatrix} \dfrac{AA'}{A+A'} & \dfrac{AB'+A'B}{A+A'} \\ \dfrac{A'C+AC'}{A+A'} & D+D'+\dfrac{B'C+BC'-BC-B'C'}{A+A'} \end{pmatrix}. \tag{7-1}$$

For parallel inputs and parallel outputs,

$$\boldsymbol{T}_{\mathrm{pipo}} = \begin{pmatrix} \dfrac{A'B+AB'}{B+B'} & \dfrac{BB'}{B+B'} \\ C+C'+\dfrac{AD'+A'D-AD-A'D'}{B+B'} & \dfrac{BD'+B'D}{B+B'} \end{pmatrix}. \tag{7-2}$$

If instead the inputs are in series and the outputs are in series, then

$$T_{\text{siso}} = \begin{pmatrix} \dfrac{AC' + A'C}{C + C'} & B + B' + \dfrac{A'D + AD' - AD - A'D'}{C + C'} \\ \dfrac{CC'}{C + C'} & \dfrac{C'D + CD'}{C + C'} \end{pmatrix}. \tag{7-3}$$

Finally, for series inputs and parallel outputs we have

$$T_{\text{sipo}} = \begin{pmatrix} A + A' + \dfrac{BC' + B'C - BC - B'C'}{D + D'} & \dfrac{B'D + BD'}{D + D'} \\ \dfrac{CD' + C'D}{D + D'} & \dfrac{DD'}{D + D'} \end{pmatrix}. \tag{7-4}$$

These general expressions allow branches of any sort to be combined, provided of course that a mechanical port is joined only with a mechanical port, etc. If more than two branches are to be combined, one need just apply these expressions repeatedly until a branch-free cascaded system results. It is easily shown that these combination matrices are determinant preserving in the sense that if $\det T = \det T'$, then the determinant of the combination equals $\det T$. (Note: the parallel branch T-matrices given in ref. [2] p. 342 contain a typographical error, and apply only to the case $\det T = 1$.)

We note for reference that if N identical branches are connected together with parallel inputs and series outputs, the overall T-matrix is

$$T_{\text{piso}} = \begin{pmatrix} A/N & B \\ C & DN \end{pmatrix}. \tag{7-5}$$

Similarly,

$$T_{\text{pipo}} = \begin{pmatrix} A & B/N \\ CN & D \end{pmatrix} \tag{7-6}$$

$$T_{\text{siso}} = \begin{pmatrix} A & BN \\ C/N & D \end{pmatrix} \tag{7-7}$$

and

$$T_{\text{sipo}} = \begin{pmatrix} AN & B \\ C & D/N \end{pmatrix}. \tag{7-8}$$

These relationships are useful in identifying the consequences of the use of multiple transducers on a system's overall transfer or impedance characteristic.

One important practical branched system is the vented or "bass reflex" loudspeaker enclosure. In this system, the transducer's output is obtained through two acoustic branches: a direct path, and a path from the rear of the transducer through the enclosure and its vent. The T-matrix for the direct branch alone is just that of its acoustic load impedance Z_a,

$$T = \begin{pmatrix} 1 & Z_a \\ 0 & 1 \end{pmatrix}. \tag{7-9}$$

For the enclosure branch, we combine an overall minus sign (to represent the rear of the transducer) with the enclosure's acoustic admittance Y followed by the vent's impedance, Z_v:

$$T' = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \begin{pmatrix} 1 & Z_v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -Z_v \\ -Y & -1 - YZ_v \end{pmatrix}. \tag{7-10}$$

Because pressures add at the input and volume velocities add at the output of this kind of branched system, we combine the T and T' matrices according to the SIPO rule, (7-4), to find the vented enclosure's T-matrix:

$$T_{\text{BR}} = \begin{pmatrix} 1 + \dfrac{Z_a}{Z_v} & Z_a + \dfrac{1}{Y} + \dfrac{Z_a}{Z_v Y} \\ \dfrac{1}{Z_v} & 1 + \dfrac{1}{YZ_v} \end{pmatrix}. \tag{7-11}$$

To install a loudspeaker into this enclosure, we need only multiply T_{BR} onto the right of the loudspeaker's T-matrix, expression (4-2), to find the electroacoustic end-to-end characteristic:

$$T_{\text{BRLS}} = \begin{pmatrix} \dfrac{RS}{Bl} + \dfrac{RZ_m}{SBlZ_v} + \dfrac{Bl}{SZ_v} & \dfrac{RS}{BlY} + \dfrac{RZ_m}{SBl} + \dfrac{RZ_m}{SBlYZ_v} + \dfrac{Bl}{S} + \dfrac{Bl}{SYZ_v} \\ \dfrac{S}{Bl} + \dfrac{Z_m}{SBlZ_v} & \dfrac{S}{BlY} + \dfrac{Z_m}{SBl} + \dfrac{Z_m}{SBlYZ_v} \end{pmatrix}. \tag{7-12}$$

In this expression, the driver's acoustic load Z_a has been absorbed into its mechanical impedance, Z_m, as in expression (4-3). The electrical input impedance of the system is the ratio B/D, from eq. (2-6). From eq. (4-3), the sensitivity for given voltage excitation is $u_2/e_1 = 1/B$. The sound pressure which the system produces in free space at distance d can be obtained with the help of expressions (6-1) or (6-6). The system's response can be put into a more explicitly frequency-dependent form by substituting the appropriate frequency-dependent functions for the admittances and impedances appearing in eq. (7-12), i.e. $j\omega V/\gamma P_0$ for Y, $j\omega\varrho_0 L/S_v$ for Z_v, and so forth. With these substitutions, one obtains the usual bass reflex response functions discussed more fully in refs. [19] and [20].

8. Applications

The mathematical technique outlined here appears to have three applications to everyday electroacoustic work. The first application is the obvious one of algebraic analysis of given systems. In this regard, T-matrices combine to give expressions for end-to-end transfer functions in a particularly straightforward manner. As an instructional aid, the T-matrix approach offers a clear and easily grasped explanation of how a signal is conveyed through a cascaded system and of how the various network stages together define the overall transfer characteristic. Adding or removing stages from the system becomes, mathematically speaking, especially simple.

A second application is to the numerical analysis of linear systems by digital computer. One frequently wishes to calculate and display graphs of a system's response and impedance functions for various values of adjustable parameters. Programs of this kind are usually based on evaluating a function having a fixed algebraic form, worked out in advance by the engineer; they apply only to a certain fixed sequence of circuit elements. The T-matrix method allows more versatile modular codes to be created. One could, for instance, prepare a subroutine for each type of network stage, and let the sequence in which the subroutines are called define the configuration of the system. Such a code would be a particular benefit in interactive design work, where the system's configuration must itself be regarded as an important design variable.

A third application is to the synthesis of linear systems having some desired response characteristic. Here, one has a specified overall T-matrix and one or more given component stages, and wishes to fill in the remainder of the system. This can be accomplished using the factorization procedure given in section 2, which in essence expresses the unknown desired matrix as the ratio of the overall T-matrix to the given subsystem's matrix. Examples are found in the design of crossover networks, equalizers, and the like.

Acknowledgment

The author wishes to express his thanks for many stimulating and useful discussions with Mr. J. Henry Primbsch, Dr. Susan M. Lea, and Prof. W. J. Welch.

Appendix

The existence of T, Y, and Z matrices

As discussed in section 2 above, a linear source-free two-port network must satisfy two equations in the four variables e_1, i_1, e_2, and i_2. The most general pair of homogeneous linear equations in these variables is

$$\begin{aligned} E\,e_1 + F\,i_1 + G\,e_2 + H\,i_2 &= 0 \\ J\,e_1 + K\,i_1 + L\,e_2 + M\,i_2 &= 0\,. \end{aligned} \tag{A-1}$$

In these terms, the $\boldsymbol{T}$ matrix is

$$\boldsymbol{T} = \begin{pmatrix} \dfrac{FL-GK}{EK-FJ} & \dfrac{FM-HK}{EK-FJ} \\ \dfrac{GJ-EL}{EK-FJ} & \dfrac{HJ-EM}{EK-FJ} \end{pmatrix} \tag{A-2}$$

which exists if $EK-FJ \neq 0$. If the two-port network is nonlateral, i.e. it has no connection between its ports, then eq. (A-1) splits into two uncoupled port expressions, $EK-FJ$ is zero, and no T-matrix representation is possible. The input cannot, in this case, be inferred from the output variables. T exists whenever the output is affected by the input.

Whenever $\boldsymbol{T}$ exists, its reciprocity index is $\det T = (GM-HL)/(EK-FJ)$.

The impedance matrix may be calculated from (A-1):

$$\boldsymbol{Z} = \begin{pmatrix} \dfrac{GK-FL}{EL-GJ} & \dfrac{HL-GM}{EL-GJ} \\ \dfrac{FJ-EK}{EL-GJ} & \dfrac{EM-HJ}{EL-GJ} \end{pmatrix} \tag{A-3}$$

which exists if $EL-GJ \neq 0$. All series circuits violate this criterion, because for them one of the equations in (A-1) must be $i_1 = i_2$ and hence either $E=G=0$ or $J=L=0$; the voltages cannot be inferred from the currents alone.

Similarly, the Y matrix may be calculated:

$$Y = \begin{pmatrix} \dfrac{HJ - EM}{FM - HK} & \dfrac{HL - GM}{FM - HK} \\ \dfrac{FJ - EK}{FM - HK} & \dfrac{FL - GK}{FM - HK} \end{pmatrix} \qquad \text{(A-4)}$$

which exists if $FM - HK \neq 0$. All shunt circuits violate this criterion, because for them one of the eqs. (A-1) must be $e_1 = e_2$ and hence either

$$F = H = 0 \quad \text{or} \quad K = M = 0;$$

in either case, the currents cannot be inferred from the voltages.

The relationships between the T, Z, and Y matrices are these:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} Z_{11}/Z_{21} & \det Z/Z_{21} \\ 1/Z_{21} & Z_{22}/Z_{21} \end{pmatrix} = \\ = \begin{pmatrix} -Y_{22}/Y_{21} & -1/Y_{21} \\ -\det Y/Y_{21} & -Y_{11}/Y_{21} \end{pmatrix} \qquad \text{(A-5)}$$

$$\begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} = \begin{pmatrix} A/C & \det T/C \\ 1/C & D/C \end{pmatrix} = \qquad \text{(A-6)} \\ = \begin{pmatrix} Y_{22}/\det Y & -Y_{12}/\det Y \\ -Y_{21}/\det Y & Y_{11}/\det Y \end{pmatrix}$$

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} D/B & -\det T/B \\ -1/B & A/B \end{pmatrix} = \qquad \text{(A-7)} \\ = \begin{pmatrix} Z_{22}/\det Z & -Z_{12}/\det Z \\ -Z_{21}/\det Z & Z_{11}/\det Z \end{pmatrix}.$$

(Received May 11th, 1977.)

References

[1] Huelsman, L. P., Circuits, matrices, and linear vector spaces. McGraw-Hill, New York 1963, p. 57.
[2] Pipes, L. A., Matrix methods in engineering. Prentice-Hall, 1963, pp. 342–343.
[3] Reference data for radio engineers (Ed. H. P. Westman). ITT, New York 1956, pp. 660–663.
[4] Molloy, C. T., Four pole parameters in vibration analysis. Colloq. on Mechanical Impedance Methods for Mechanical Vibrations (Ed. R. Plunkett). Appl. Mech. Div., ASME; 1958, pp. 43–68.
[5] Rocke, R. D., Transmission matrices and lumped parameter models for continuous systems. Ph. D. thesis, Caltech, Pasadena 1966.
[6] Mitra, S. K., Analysis and synthesis of linear active networks. J. Wiley, New York 1969, pp. 22–82.
[7] McMillan, E. M., Violation of the reciprocity theorem in linear passive electromechanical systems. J. Acoust. Soc. Amer. 18 [1946], 344.
[8] Hunt, F. V., Electroacoustics. Harvard U. Press, Cambridge 1954, pp. 92–95 and 103–116.
[9] Guillemin, E. A., Synthesis of passive networks. Wiley, New York 1957, pp. 5–7.
[10] Newcomb, R. W., On network realizability conditions. Proc. IRE 50 [1962], 1995.
[11] Woods, D., Reappraisal of the unconditional stability criteria for active two-port networks in terms of S parameters. IEEE Trans. CAS-23 [1976], 73.
[12] Bauer, B. B., Equivalent circuit analysis of mechano-acoustic structures. IRE Trans. AU-2 [1954], 112, reprinted in J. Audio Eng. Soc. 24 [1976], 643.
[13] Beranek, L. L., Acoustics. McGraw-Hill, New York 1954, pp. 70–72.
[14] Olson, H. F., Acoustical engineering. D. Van Nostrand, New Jersey 1957, p. 92.
[15] Mason, W. P., Electromechanical transducers and wave filters. D. Van Nostrand, New Jersey 1942, p. 187.
[16] Miles, J. W., Coordinates and the reciprocity theorem in electromechanical systems. J. Acoust. Soc. Amer. 19 [1947], 910.
[17] Frazer, R. A., Duncan, W. J., and Collar, A. R., Elementary matrices, Cambridge University Press, Cambridge 1952, pp. 39–41.
[18] MacLean, W. R., Absolute measurement of sound without a primary standard. J. Acoust. Soc. Amer. 12 [1940], 140.
[19] Lea, S. M. and Lampton, M. L., Theory of maximally flat loudspeaker systems. IEEE Trans. AU-20 [1972], 200.
[20] Small, R. H., Vented box loudspeaker systems. J. Audio Eng. Soc. 21 [1973], 363.