The Theory of Maximally Flat Loudspeaker Systems

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Abstract

A convenient mathematical formulation is given for the absolute low-frequency response of an electrodynamic direct radiator loudspeaker mounted in a vented or unvented enclosure. The combined effects of speaker Q (ratio of reactance to resistance), enclosure size, and enclosure tuning are shown to control the convergence of the response curve towards ts asymptotic value. Mathematical criteria are presented graphically, which enable the designer to find enclosure characteristics that give a frequency response curve flat through fourth order in reciprocal frequency, for a given loudspeaker Q. In addition, it is shown that for $Q=0.383$ the system can be made flat through sixth order by choosing the correct enclosure size and vent tuning. This "triple point" design represents the flattest possible response for direct radiator systems.

I. Introduction

The frequency response of an electrodynamic direct radiator loudspeaker mounted in an enclosure has been discussed by many authors [1]–[5]. The considerable mathematical complexity of theoretical treatments has, however, largely discouraged engineers from attempting to mathematically optimize the performance of such systems. It is the purpose of this paper to give a useful formulation of the response of a loudspeaker system, and to show how its response curve can be made as flat as the designer's constraints permit.

II. Theory

Let the dc resistance of the voice coil of the loudspeaker be denoted as R. We define the absolute response $E(\omega)$ as the ratio of the radiated acoustic power for some driving voltage V_0 to the electrical power that would be dissipated in a resistor R at the same voltage. With this definition, $E(\omega)$ is a measure of the system's frequency response with constant voltage drive. If needed, the system efficiency can be obtained by dividing the response by R and by the real part of the system's complex electrical admittance.

$E(\omega)$ can be derived from the presentations given by Olson [4, p. 126], Novak [5], or from the following considerations. Let $z_1=f_1/v_1$ represent the complex mechanical impedance of the piston and its air load, and let r_{em} denote the electrodynamic drag of the voice coil/magnet assembly $(BL)^2/R$. Then from Ohm's law, $V_0-BLv_1=IR=f_1R/BL$. The resulting piston velocity is then given by

$$v_1 = V_0 \frac{BL/R}{z_1 + r_{em}}.$$

When the radiated acoustic power $v_1^2 r_{\rm rad}$ is divided by V_0^2/R, there results

$$E(\omega) = \frac{r_{em} r_{\rm rad}}{|z_1 + r_{em}|^2} \, |F|^2 \tag{1}$$

in which F represents the ratio of the vent plus piston volume velocity to that of the piston alone. Sealed enclosures have $F=1$.

As is customary, we shall take the radiation resistance of the diaphragm to be its low-frequency limiting form, given by $\rho A^2\omega^2/2\pi c$, where A represents the piston area, ρ the sea level density of air, and c the speed of sound. This expression is valid for all wavelengths substantially greater than the diameter of the piston. For F we take the expression $\omega^2/(\omega^2-\omega_h^2)$ where the square of the raidan Helmholtz frequency $\omega_h^2=s_2/m_2$; s_2 represents the compressional stiffness of the air in the enclosure, and m_2 the effective mass of the air in the vent. Sealed enclosures have $m_2=\infty$.

The complex expression for z_1 is

$$z_1 = i\omega m_1 + r_1 + \frac{s_1}{i\omega} + F\frac{s_2}{i\omega} + r_{\rm rad} \tag{2}$$

in which m_1 represents the total effective mass of the loudspeaker piston including its air load, r_1 the mechanical resistance of the cone suspension, and s_1 the suspension stiffness or spring constant. The frequency dependent radiation resistance term in z_1 may be safely neglected, because at all frequencies it is dominated by one or another of the terms in z_1+r_{em}. We have not included the small reactive cross impedance terms arising from external acoustic coupling between the diaphragm and the vent [6], [7].

We find it convenient to deal with dimensionless numbers wherever possible. Following Novak [5], let $g=\omega/\omega_1$ be the ratio of the driving frequency to the free air resonant frequency of the loudspeaker; thus with the stiffness and mass defined above, $\omega_1^2=s_1/m_1$. The short circuit Q value of the loudspeaker is given by its ratio of mechanical reactance to mechanical resistance, or $(s_1m_1)^{\frac{1}{2}}/(r_{em}+r_1)$. Finally, we shall define the dimensionless enclosure tuning ratios $S=s_2/s_1$ and $M=m_2/m_1$. In terms of these quantities, (1) and several pages of algebra lead to

$$E(g) = \frac{r_{em}\rho A^2}{2\pi c m_1^2} \cdot \frac{1}{1 + Ag^{-2} + Bg^{-4} + Cg^{-6} + Dg^{-8}} \tag{3}$$

where

$$A = \frac{1}{Q^2} - 2 - 2S - 2\frac{S}{M} \tag{3.1}$$

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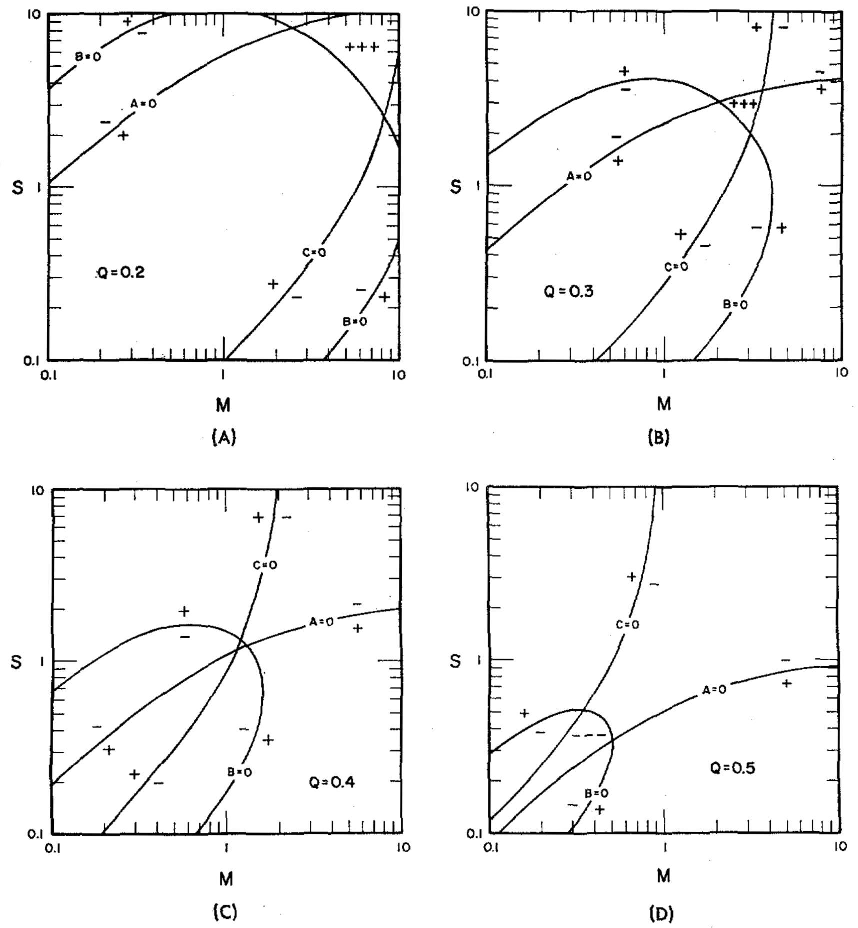

Fig. 1. Solutions of the equations $A=0$, $B=0$, and $C=0$ for values of $Q=0.2$, 0.3, 0.4, and 0.5.

$$B = 1 + 2S + S^2 + 4\frac{S}{M} + 2\frac{S^2}{M} + \frac{S^2}{M^2} - 2\frac{S}{Q^2M} \quad (3.2)$$

$$C = \frac{S^2}{Q^2M^2} - 2\frac{S}{M} - 2\frac{S^2}{M} - 2\frac{S^2}{M^2} \quad (3.3)$$

$$D = \frac{S^2}{M^2}. \quad (3.4)$$

The first thing to notice about (3) is that as the frequency g increases, the second factor approaches unity. The asymptotic efficiency is then given by the first factor, which is the well-known quantity $r_{em}\rho A^2/2\pi c\ m_1^2$. The shape of the response curve is due entirely to the second factor, which contains only the dimensionless quantities Q, S, and M appearing in (3.1)–(3.4). We shall now turn to an investigation of the behavior of this second factor.

It is clear that the frequency response function (3) would be absolutely flat if we could arrange to have $A=B=C=D=0$. In fact, these four conditions cannot be simultaneously satisfied by the three quantities Q, S, and M. It follows that no loudspeaker described by (1) can be absolutely flat. We can, however, require that one, two, or three of these coefficients vanish by adjusting one, two, or three of the parameters Q, S, and M. As the frequency g is reduced, it is the term involving A that first affects the response curve. Further reductions in the driving frequency cause the succeeding terms B, C, and D to become significant. A maximally flat system is one in which as many of these terms as possible have been made zero, starting with A. Note, for example, that the condition $B=0$ will not be of any particular help in achieving flat response if A is not zero. Before discussing the design of maximally flat systems, however, we shall first examine the separate conditions $A=0$, $B=0$, and $C=0$.

III. The General Case

Because we are interested in the dependence of the coefficients A, B, and C on the parameters Q, S, and M, we shall graphically display the loci of the roots of $A=0$, $B=0$, and $C=0$ in the (S, M) plane for several values of Q. In Fig. 1(A)–(D) we have sketched these root loci for $Q=0.2$, 0.3, 0.4, and 0.5, which spans the range of commonly available loudspeakers. Adjacent to each root locus, a $\pm$ symbol indicates the sign of A, B, or C. The regions of the S, M plane marked $+++$ or $---$ are the zones where A, B, and C are all positive or all negative. In general, the three curves do not intersect at a common point.[1]

[1] *Note Added in Proof:* An alternative and fruitful treatment of loudspeaker response in terms of high-pass electrical filters has been presented by Thiele [10]. The utility of the present treatment is made clear in Fig. 1, where the separate effects of changes in Q, S, and M on each of the response polynomial coefficients can be visualized.

IV. The Condition $A=0$

This condition will guarantee that the response curve (3) converges towards its asymptotic value faster than g^{-2}. Such a system can be described as being flat through second order in reciprocal frequency. Equation (3.1) reveals that this condition can be satisfied for any loudspeaker having $Q<0.707$ provided that S or M is chosen to give $S+S/M+1=1/(2Q^2)$. Sealed enclosure "air suspension" systems can satisfy this condition, provided that the stiffness ratio $S=1/(2Q^2)-1$. For such systems the 3 dB frequency is given by $[(s_1+s_2)/m_1]^{\frac{1}{2}}$ rad/s.

In Fig. 2, we illustrate the contours of S versus M for which $A=0$, for several values of Q. Tuning ratios that lie above the appropriate Q curve are associated with negative values of A. This means that the response approaches its asymptotic value from above. Such systems can be described as peaky, since they have one or more maxima in $E(g)$. A system whose (S, M) point lies below its $A=0$ curve will show a second order asymptotic approach from below. Because the three parameters Q, S, and M are available to the engineer, there is a considerable range of possible designs for which A can be zero.

V. The Condition $A=B=0$

When this condition is satisfied, the response can be described as flat through fourth order in reciprocal frequency. Equations (3.1) and (3.2) must be simultaneously satisfied by Q, S, and M in order to achieve this condition. Since we have two equations in three unknowns, any one of the variables can be specified in advance and the remaining quantities determined by $A=B=0$. Analysis reveals that valid solutions exist for all $Q<0.563$ and $M<3$. The fact that no valid solutions exist for larger values of M implies that sealed enclosure "air suspension" systems are not flat through fourth order.

In Fig. 2, the heavy curve gives the locus of (S, M) values having $A=B=0$ as a function of Q. From this curve, the choice of enclosure parameters S and M can be made that maximizes the flatness of response for a loudspeaker with given Q.

VI. The Condition $A=B=C=0$

To satisfy this condition, the formulas (3.1), (3.2), and (3.3) must be solved for Q, S, and M. One solution exists, with

$$Q = [\tfrac{1}{2} - (\tfrac{1}{8})^{\frac{1}{2}}]^{\frac{1}{2}} = 0.383 \quad (4.1)$$

$$S = 2^{\frac{1}{2}} = 1.414 \quad (4.2)$$

$$M = 2^{\frac{1}{2}} = 1.414. \quad (4.3)$$

This triple point describes the system governed by (3) which

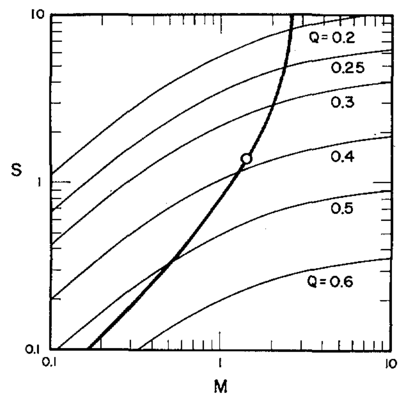

Fig. 2. Loci of (S, M) values for which $A=0$, for several values of Q. Heavy line represents solutions of $A=B=0$; circle represents the triple point $A=B=C=0$.

has the flattest possible response. Any other choice of Q, S, or M will produce a frequency response curve that converges towards its asymptote more slowly than this triple point convergence. For this system $D=1$, and the low-frequency output is down 3 dB at $g=1$ or $\omega=\omega_1$. It is then the loudspeaker's free air resonance that sets the low-frequency limit to the performance of maximally flat systems.

We note in passing that when $A=B=C=0$, (3) reduces to the transfer function of a Butterworth filter [8] having four poles. This identification suggests that designs analogous to Chebychev filters should also be possible, where A, B, and C are small and alternate in sign, as they do in the $-+-$ regions of Fig. 1(A)–(D). Chebychev filters are not maximally flat, but can be designed to exhibit acceptably small pass-band ripple.

VII. Evaluation

The designs resulting from applying the conditions $A=B=0$ and $A=B=C=0$ have been evaluated numerically, through the use of a general purpose computer program [9] that calculates system response and impedance functions and includes small corrections to (3) arising from such a phenomenon as radiation damping. The results of these evaluations are graphed in Fig. 3, where a single loudspeaker with piston area of $0.032m^2$, suspension stiffness $s_1=1000$ N/m, and free air resonant frequency of 30 Hz has been assigned various BL products to obtain Q values of 0.30, 0.383, and 0.45. In each case, the enclosure parameters S and M were chosen to satisfy $A=B=0$. Other details of these systems are listed in Table I. All the resulting curves show rapid convergence toward the asymptotic efficiency. As would be expected from studying Fig. 1(B) and (D), the cases of $Q<0.383$ and $Q>0.383$ show convergence from below and from above, respectively; this fact is the consequence of the positive and negative values of C in these cases. When $Q=0.383$, the convergence is extremely rapid, which is characteristic of the triple point maximally flat system.

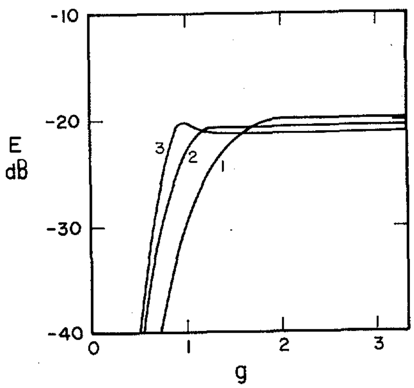

Fig. 3. Absolute response of three systems for which $A=B=0$. Curve 2 also satisfies $C=0$.

TABLE I

Characteristics of the Systems in Fig. 3

Parameter	Curve 1	Curve 2	Curve 3
Volume, m^3	0.048	0.103	0.209
BL, Wb/m	11.3	9.8	9.0
Q	0.300	0.383	0.450
S	3.03	1.414	0.695
M	1.99	1.414	0.898

VIII. Conclusions

We conclude that a loudspeaker system designed in accord with the theoretical guidelines given here does indeed have exceedingly flat response. The system can be made flat to any required frequency by choosing a speaker with a suitable free air resonance. A practical restraint is that high-compliance speakers will require large enclosures in order to give optimum response. By choosing a lower compliance, higher mass speaker, this problem can be alleviated.

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