Hyperbolic Lens

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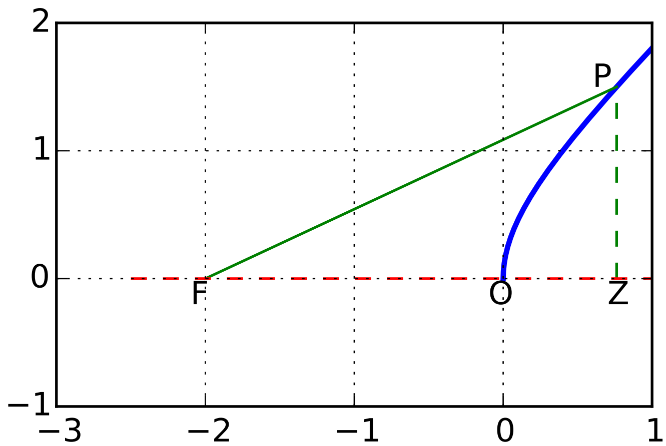
The equation of a hyperboloid with one vertex at the origin in cylindrical coordinates is

$$z = \frac{Cr^2}{1 + \sqrt{1 - SC^2 r^2}} \tag{1}$$

where C is the vertex curvature and S is a shape parameter, $S < 0$ for hyperboloids. Invert this, solving for r^2 as a function of z:

$$r^2 = 2z/C - z^2 S \tag{2}$$

The blue curve in the figure is a hyperbola. If F is a perfect focus, then every ray FP must



satisfy the principle of stationary phase: its optical path must equal the axial optical path to that same z. From the theorem of Pythagoras, the optical path of the ray is $\sqrt{(F+z)^2 + r^2}$ while the optical path along the axis to the same z is $F + nz$ where n is the refractive index of the convex lens. Equating, squaring, and solving:

$$r^2 = 2zF(n-1) + z^2(n^2 - 1) \tag{3}$$

which is the same as the hyperbola provided that $C = 1/(n-1)F$ which is obeyed by all convex lenses, and provided that $S = 1 - n^2$.

Q.E.D.

References:

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