Note on the Accelerated Charge Paradox

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It is well known that an accelerated charge radiates. A charge supported motionless in a uniform static gravitational field has no acceleration and therefore cannot radiate. It is equally well known that a uniform static gravitational field is equivalent to a uniformly accelerated coordinate frame: they cannot be distinguished by any means. This is Einstein's equivalence principle. So: may we simply look at a charge, and measure its radiation, and thereby distinguish if we are accelerating or gravitating?

There is a bewildering amount of literature on this question.

Let's start with Larmor's formula, conventionally derived by integrating the outgoing Poynting power flux[1] for nonrelativistic acceleration a:

$$P_{rad}(t) = \frac{2e^2}{3c^3}\, a^2 \tag{1}$$

or, for relativistic motion, Lienard's formula[2] gives

$$P_{rad}(t) = \frac{2}{3}\,\frac{e^2}{c}\,\gamma^6\, [(\dot\beta^2 - (\beta \times \dot\beta)^2] \tag{2}$$

which for relativistic straight-line motion with momentum p reduces to[3]

$$P_{rad} = \frac{2}{3}\,\frac{e^2}{m^2c^3}\,\dot p^2 \tag{3}$$

An important historical approach[4] was the Abrahan-Lorentz-Dirac equation of motion that introduced a third derivative of position, or *superacceleration*:

$$m\ddot x - m\tau_e \dddot x = f(t) \tag{4}$$

Here, τ_e is that tiny time delay, $2e^2/3mc^3 = 6 \times 10^{-24}$s, needed for light to propagate across the classical electron radius. The ALD formula is repugnant because it admits an unphysical solution: the tiniest nudge to an electron will make $\ddot x$ and $\dddot x$ have the same sign and have a positive dot product, and so run away with no applied force, violating the conservation laws for both momentum and energy. This unphysical behavior has been eliminated by Spohn[5] and Rohrlich[6] by correcting the ALD equations.

[1] Jackson J.D. "Classical Electrodynamics" Third Edition, eq 14.22 (1999)
[2] ibid., eq 14.26
[3] ibid., eq 14.27
[4] Dirac,P.A.M., "Classical Theory of Radiating Electrons,"Proc Roy Soc v.167 (1938)
[5] Spohn, H., "Critical manifold of the Lorentz-Dirac equation," Europhys Lett v.50 (2000)
[6] Rohrlich, F., "Correct equation of motion of a classical point charge," Phys Lett A v.283 (2001)

Richard Feynman explained[7] that this paradox is based on a misconception: the Larmor formula is valid only for circular motion, e.g. in synchrotrons, and is not general:

> MODIFICATIONS OF ELECTRODYNAMICS REQUIRED BY THE PRINCIPLE OF EQUIVALENCE
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> "The Principle of Equivalence postulates that an acceleration shall be indistinguishable from gravity by any experiment whatsoever. In particular, it cannot be distinguished by observing electromagnetic radiation. There is evidently some trouble here, since we have inherited a prejudice that an accelerating charge should radiate, whereas we do not expect a charge lying in a gravitational field to radiate. This is, however, not due to a mistake in our statement of equivalence but to the fact that the rule of power radiated by an accelerating charge,
>
> $$dW/dt = 2e^2a^2/3c^3, \tag{5}$$
>
> has led us astray. This is usually derived from calculating the flow from Poynting's theorem far away, and it is only valid for cyclic motions, or at least motions which do no grow forever in time (as a constant acceleration does). It does not suffice to tell us *when* the energy is radiated. This can only be determined by finding the force of radiation resistance, which is $(2/3)(e^2/c^3)\dot{\vec{a}}$. For it is work against this force which represents energy loss. For constant acceleration this force is zero. Generally the work done against it can be written
>
> $$\frac{dW}{dt} = -\frac{2}{3}\frac{e^2}{c^3}\vec{v}\cdot\dot{\vec{a}} \tag{6}$$
> $$= \frac{2}{3}\frac{e^2}{c^3}\vec{a}\cdot\vec{a} - \frac{2}{3}\frac{e^2}{c^3}\frac{d}{dt}\left(\vec{v}\cdot\vec{a}\right) \tag{7}$$
>
> giving a correct expression for dW/dt."

Feynman's general expression (his eqn 9.2, my eqn 6 above) gives positive outgoing radiation for circular motion since $\vec{v}\cdot\dot{\vec{a}} < 0$. Synchrotrons radiate, so this is good. However Rowland[8] reminds us that that linear accelerations allow $\vec{v}\cdot\dot{\vec{a}} > 0$ for which the radiated power is negative. Rowland's example demonstrates a linear runaway, every bit as troubling as the ALD runaway.

[7] Feynman Lectures on Gravitation, Caltech, 1962-1963, excerpt pp 123-124, reprinted in https://forums.futura-sciences.com/physique/8103

[8] Rowland D. R., "Comment on Poynting Flux," Eur. J. Phys. v.39 (2018)

Another approach to understanding this situation is to reexamine the dynamics of an accelerated charge. A key fact is that a point charge cannot exist. By considering an enlarged charge, the infinities can be removed. One must take care to be consistent in the order of expansion. For example R.F.O'Connell ("The equation of motion of an electron," Phys.Lett A 313, 2003, equation 3) presented, with again $\tau_e = 2e^2/3mc^3 = 6 \times 10^{-24}s$:

$$m\ddot{x} = f(t) + \tau_e \dot{f}(t) \tag{8}$$

from which he obtains the radiated power:

$$P_{rad}(t) = \frac{\tau_e}{M} f^2(t) = \frac{2e^2}{3c^3} \frac{f^2(t)}{M^2} \tag{9}$$

which differs from the Larmor power by terms of only τ_e^2.

Does O'Connell's electron run away?