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CENTER GRAPHS OF CHORDAL GRAPHS

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ABSTRACT. A graph G is chordal (triangulated, rigid circuit) if every cycle of length at least 4 has a chord. The center $C(G)$ of any graph G is the set of vertices of G with the minimum eccentricity, and $<C(G)>$, the subgraph of G induced on $C(G)$, is the center graph of G. A graph G is said to be self-centered if $C(G) = V(G)$ or equivalently $r(G) = d(G)$ where $r(G)$, $d(G)$ denote the radius, diameter, respectively, of G. Laskar and Shier (Abstracts AMS (1981), 783-05-14) proved, among other results, that if G is a self-centered chordal graph, then $r(G) \leqslant 3$. In this paper we show that there are no self-centered chordal graphs with $r(G) = 3$ and characterize all self-centered chordal graphs. All chordal graphs H with $(r(H), d(H)) \neq (2, 3)$ which are the center graphs of some chordal graph are also characterized. For a connected chordal graph G, let $a(G) = k$ be the smallest positive integer k such that $C^k(G) = C^{k-1}(G)$ where $C^\circ(G) = V(G)$, $C^i(G) = C(<C^{i-1}(G)>)$. Then we show that $a(G) \leqslant 4$ and the bound is sharp.

1. Introduction and definitions

For an undirected graph G without loops or multiple edges, the symbols $V(G)$, $E(G)$ denote the vertex set, edge set, respectively of G; and $|V(G)|$ is the *order* of G. A graph G is called *chordal* (triangulated, rigid circuit) if every cycle of length at least 4 has a chord, that is, an edge joining non-consecutive vertices of the cycle. Examples of chordal graphs include k-trees, interval graphs and block graphs. Further, chordal graphs are known to be perfect [1] and efficient algorithms exist for finding minimum colorings, minimum independent sets, minimum clique covers and maximal cliques in chordal graphs (refer to Gavril [6], Golumbic [7]); even though these problems are known to be NP-hard for general graphs [7]. Chordal graphs also arise in several application areas which include, among others, evolutionary trees [3], facility location [4], scheduling [11] and solution of sparse systems of linear equations [13].

Let $\mathcal{G}$ be the set of all non-isomorphic types of connected chordal graphs. A vertex $v \in V(G)$ is said to be *simplicial* if its neighbourhood $N_G(v) = \{x \in V(G)/vx \in E(G)\}$, induces a complete graph. Let $S(G)$ be the set of all simplicial vertices of G. A subset $S \subseteq V(G)$ is a *vertex separator* of G for non-adjacent vertices a and b (or an a—b separator of G) if in $G—S$, the graph obtained from G by the removal of the vertices of S and incident edges, the vertices a and b belong to distinct connected components. Let $S(a, b, G)$

be the set of all a—b separators of G. If no proper subset of $S \in S(a, b, G)$ belongs to $S(a, b, G)$, then S is called a *minimal* a—b separator of G. Let $S_0(a, b, G)$ be the set of all minimal a—b separators of G.

A clique C of G is a subset $C \subseteq V(G)$ such that $<C>$, the subgraph of G induced on the set C is complete in G. A clique C is said to be an *absorbent clique* in G if for every $u \in V(G)-C$ there exists a $v \in C$, such that $uv \in E(G)$. Let G be a connected graph. Let $x, y \in V(G)$.

$$d_G(x, y) = \text{distance between } x \text{ and } y \text{ in } G$$

$$\text{eccentricity of } x,\ e_G(x) = \underset{z \in V(G)}{\text{maximum}}\ d_G(x, z)$$

$$\text{radius of } G,\ r(G) = \underset{z \in V(G)}{\text{minimum}}\ e_G(z)$$

$$\text{diameter of } G,\ d(G) = \underset{z \in V(G)}{\text{maximum}}\ e_G(z).$$

$N_G^k(x) = \{y \in V(G)/d_G(x, y) = k\}$ where k is a positive integer.

Define $$C(G) = \{x \in V(G)/e_G(x) = r(G)\}.$$

$C(G)$ is called the *center* of G, and $<C(G)>$ is the *center graph* of G. A graph G is said to be *self-centered* if $r(G) = d(G)$, equivalently $C(G) = V(G)$. Laskar and Shier [9] proved that if $G \in \mathcal{G}$ and is self-centered, then $r(G) \leqslant 3$. In this paper we prove that if $G \in \mathcal{G}$ and is self-centered, then $r(G) \leqslant 2$ [Theorem 12] and also characterize all self-centered $G \in \mathcal{G}$. [Theorem 15]. Laskar and Shier [9] proved that if $G \in \mathcal{G}$, then $<C(G)>$ is connected. We show that if $G \in \mathcal{G}$ then either $C(G)$ is complete or $k(G) \leqslant k(<C(G)>)$ where $k(G)$ is the vertex connectively of G [Corollary 4]. Let $G \in \mathcal{G}$ and $a = a(G)$ be the smallest positive integer such that $<C^a(G)> = <C^{a-1}(G)>$ where $C^0(G) = V(G)$ and $C^i(G) = C(<C^{i-1}(G)>)$, $i \geqslant 1$. Laskar and Shier [9] proved that $a(G)$ exists for $G \in \mathcal{G}$. We prove that if $G \in \mathcal{G}$, then $a(G) \leqslant 4$ and the bound is sharp [Theorem 11]. We further characterize all $H \in \mathcal{G}$ with $(r(H), d(H)) \neq (2, 3)$ which are center graphs of some chordal graph [Theorems 15, 21].

Define
$$r_i(G) = r(<C^i(G)>)$$
$$d_i(G) = d(<C^i(G)>)$$
$$r_0(G) = r(G) \text{ and } d_0(G) = d(G).$$

We show that if $G \in \mathcal{G}$, then $d_1(G) \leqslant 3$, [Theorem 8].

The following well known results on chordal graphs are used repeatedly in the proofs of theorems and lemmas of this paper.

(1) Every chordal graph $G \neq K_n$, the complete graph of order n, has two nonadjacent simplicial vertices, Dirac [5].

(2) G is chordal iff $G-v$ is chordal for some simplicial vertex v of G.

(3) G is chordal iff for every pair, a, b of nonadjacent vertices, $a \neq b$, and $S_0 \in S_0(a, b, G)$, $<S_0>$ is complete, Dirac [5].

For notation and definition not given here, we follow, Bondy and Murty [2] and Golumbic [7].

We first prove the following basic lemma regarding members of $S_0(x, y, G)$ where $x, y \in C(G)$ and $G \in \mathcal{G}$.

Lemma 1 : *If $G \in \mathcal{G}$, x, y are distinct nonadjacent vertices in $C(G)$, and $S_0 \in S_0(x, y, G)$, then the following conditions hold*

(C1) $S_0 \subsetneq C(G)$.

(C2) *There are at least 2 distinct vertices $z_1, z_2 \in S_0$ such that for every $i = 1, 2$ either $d_G(x, z_i) = 1$ or $d_G(y, z_i) = 1$. In particular $|S_0| \geqslant 2$.* ... (1.1)

Proof : Suppose that the lemma is not true. Choose a vertex $z^* \in S_0$ as follows : If (C1) is not true, then choose a vertex $z^* \in S_0$ with $e_G(z^*) > r(G)$. If (C1) is true but (C2) is not true, then the number of vertices z in S_0 which satisfy (1.1) is at most one; choose z^* to be the vertex satisfying (1.1) if any, otherwise choose any vertex z^* in S_0.

Let a be a vertex of G with $d_G(z^*, a) = e_G(z^*)$. Without loss of generality assume that a is not in the connected component of $G-S_0$ containing x and let $P(x, a)$ be an $x-a$ geodesic in G. Notice that for any vertex w in $V(P(x, a))$

$$e_G(x) \geqslant l(P(x, a)) = l(P(x, w))+l(P(w, a)), \qquad \ldots \quad (1.2)$$

where $l(P)$ denotes the length of the path P.

If $z^* \in V(P(x, a))$, then $l(P(x, z^*)) \geqslant 1$ and $l(P(z^*, a)) \geqslant e_G(z^*)$ which by (1.2) implies that $e_G(x) \geqslant e_G(z^*)+1 \geqslant r(G)+1$, which is impossible since $x \in C(G)$. Thus $z^* \notin V(P(x, a))$. Since $S_0 \in S(x, a, G)$, $V(P(x, a))$ contains a vertex w, say, of S_0. Then $w, z^* \in S_0$ and $w \neq z^*$. Since $S_0 \in S_0(x, y, G)$ and $G \in \mathcal{G}$, by Dirac's theorem, $<S_0>$ is complete and therefore $d_G(w, z^*) = 1$. This implies that

$$l(P(w, a)) \geqslant e_G(z^*)-1. \qquad \ldots \quad (1.3)$$

If $e_G(z^*) > r(G)$, then by (1.2) and (1.3) we have that $e_G(x) \geqslant e_G(z^*) > r(G)$, which is impossible since $x \in C(G)$. Thus $e_G(z^*) = r(G)$. This implies by definition of z^*, that (C1) is true and $xw \notin E(G)$ and therefore $l(P(x, w)) \geqslant 2$. Then, as $w \in V(P(x, a))$, by (1.2) and (1.3) it follows that $e_G(x) \geqslant e_G(z^*)+1>r(G)$, contradicting the fact that $x \in C(G)$. This completes the proof of the lemma. □

From the above lemma we deduce the following

Lemma 2 : *If $G \in \mathcal{G}$, x, y are distinct nonadjacent vertices in $C(G)$, then*

$$S_0(x, y, G) = S_0(x, y, <C(G)>). \qquad \dots \quad (2.1)$$

Proof : Let $S \in S_0(x, y, <C(G)>)$.

Notice that

$$X = S \cup (V(G)-C(G)) \qquad \dots \quad (2.2)$$

belongs to $S(x, y, G)$. Let $S_0 \subseteq X$ be an element of $S_0(x, y, G)$. As $x, y \in C(G)$, by (C1) of Lemma 1, $S_0 \subseteq C(G)$ and therefore

$$S_0 \in S(x, y, <C(G)>). \qquad \dots \quad (2.3)$$

By (2.2) and (2.3), $S_0 \subseteq S$. By (2.1) and (2.3) it follows that $S = S_0 \in S_0(x,y,G)$, by selection of S_0.

Therefore

$$S_0(x, y, <C(G)>) \subseteq S_0(x, y, G). \qquad \dots \quad (2.4)$$

Conversely, let $S' \in S_0(x, y, G)$, then by (C1) of Lemma 1, $S' \subseteq C(G)$. As $x, y \in C(G)$, $S' \in S(x, y, <C(G)>)$. Let $S \subseteq S'$ be an element of $S_0(x, y, <C(G)>)$. By (2.4), $S \in S_0(x, y, G)$. Since $S, S' \in S_0(x, y, G)$ and $S \subseteq S'$ we have that $S' = S \in S_0(x, y, <C(G)>)$ and therefore

$$S_0(x, y, G) \subseteq S_0(x, y, <C(G)>). \qquad \dots \quad (2.5)$$

The relations (2.4) and (2.5) imply lemma 2. □

The following Corollaries 3, 4 and 5 are easy to prove.

Corollary 3 : *If $G \in \mathcal{G}$, x, y are distinct nonadjacent vertices of $C(G)$, then*

$$S(x, y, <C(G)>) \subseteq S(x, y, G).$$

Corollary 4 : *If $G \in \mathcal{G}$, then either $<C(G)>$ is complete or the vertex connectivity function k satisfies $k(G) \leqslant k(<C(G)>)$.*

Corollary 5 : (Laskar and Shier [9]) : *If $G \in \mathcal{G}$, then $<C(G)>$ is connected.*

Corollary 6 : *If $G \in \mathcal{G}$, then $<C(G)>$ has no cut-vertex of $<C(G)>$.*

Proof : Suppose $<C(G)>$ has a cut-vertex z, say. By Lemma 2, z is also a cut-vertex of G and this contradicts (C2) of Lemma 1. □

From Corollaries 5 and 6 we have the following

Corollary 7 : (Jordan, see, p. 73 [1]) : *If G is a tree, then* $<C(G)> = K_1$ *or* K_2.

In the next Theorem we show that the diameter of the center graph of $G \in \mathcal{G}$ is universally bounded.

Theorem 8 : *If* $G \in \mathcal{G}$, *then* $d(<C(G)>) \leqslant 3$.

Proof: By Corollary 5, $G_1 = <C(G)>$ is connected. Suppose that $d(G_1) \geqslant 4$. Choose x, y in $C(G)$ such that $d_{G_1}(x, y) = d(G_1) \geqslant 4$. Notice that $N^2_{G_1}(x) \in S(x, y, G_1)$ and therefore contains an element S_0 of $S_0(x, y, G_1)$. By Lemma 2, $S_0 \in S_0(x, y, G)$. Notice that for every $z \in S_0 \subseteq N^2_{G_1}(x)$, $d_{G_1}(x, z) \geqslant 2$ and $d_{G_1}(y, z) \geqslant d(G_1) - 2 \geqslant 2$, it follows then that $d_G(x, z) \geqslant 2$ and $d_G(y, z) \geqslant 2$, contradicting (C2) of Lemma 1. $\square$

From Theorem 8 we immediately have the following

Corollary 9 : (Laskar and Shier [9]) : *If* $G \in \mathcal{G}$ *and is self-centered, then* $r(G) = d(G) \leqslant 3$.

It may be remarked that Corollary 9 was proved in [9] from certain tight bounds for $r(G)$ in terms of $d(G)$ for $G \in \mathcal{G}$.

The following lemma will be used in the proof of Theorem 11.

Lemma 10 : *If* $G \in \mathcal{G}$ *with* $(r, d) = (2, 3)$ *then* $d(<C(G)>) \leqslant 2$.

Proof: Let $x, y \in C(G)$, $x \neq y$, and $xy \notin E(G)$ and $G_1 = <C(G)>$. Then $N^1_{G_1}(x) \in S(x, y, G_1)$. By Corollary 3, $N^1_{G_1}(x) \in S(x, y, G)$. Since $x \in C(G)$, $xy \notin E(G)$ and $r(G) = 2$, we have $d_G(x, y) = 2$. If $P(x, y)$ is an $x-y$ geodesic in G, then since $N^1_{G_1}(x) \in S(x, y, G)$, the middle vertex of $P(x, y)$ belongs to $N^1_{G_1}(x)$ and therefore $V(P(x, y)) \subseteq C(G)$ and $d_{G_1}(x, y) = 2$, proving the lemma. $\square$

In the next theorem we obtain an universal and the best possible bound for $a(G)$.

Theorem 11 : *If* $G \in \mathcal{G}$ *and is of order* n, *then* $a(G) \leqslant 4$ *and the bound is the best possible for all* $n \geqslant 10$.

Proof: By Theorem 8, the only possibilities for (r_1, d_1) are $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(1, 2)$ and $(2, 3)$. If (r_1, d_1) equals one of the first four values, then $<C^2(G)> = <C(G)>$ and therefore $a(G) \leqslant 2$. If $(r_1, d_1) = (1, 2)$, then $<C^2(G)>$ is complete and $<C^3(G)> = <C^2(G)>$, so $a(G) \leqslant 3$. If (r_1, d_1) $= (2, 3)$, then by Lemma 10, (r_2, d_2) equals $(0, 0)$, $(1, 1)$, $(2, 2)$ or $(1, 2)$. If

(r_2, d_2) equals one of the first three values, then $<C^3(G)> = <C^2(G)>$ and $a(G) \leqslant 3$. Finally if $(r_2, d_2) = (1, 2)$, then $<C^3(G)>$ is complete and $<C^4(G)> = <C^3(G)>$ and therefore $a(G) \leqslant 4$. To show that the bound is the best possible for all $n \geqslant 10$, let $G(n)$ be the graph in $\mathcal{G}$ defined below.

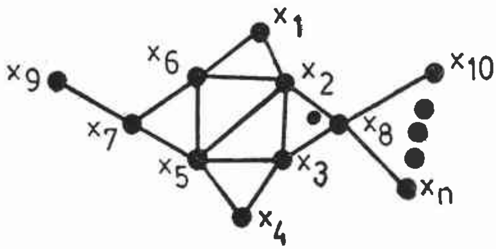
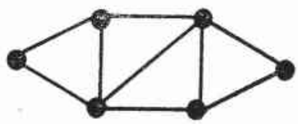


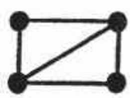
Fig. 1. The graph G(n).

Then

$$e_G(x) = \begin{cases} 3, & 1 \leqslant i \leqslant 6, \\ 4, & i = 7, 8, \\ 5, & 9 \leqslant i \leqslant n. \end{cases}$$

Consequently

$\langle C(G) \rangle =$  Fig. 2.

$\langle C^2(G) \rangle =$  Fig. 3.

$\langle C^3(G) \rangle =$  $= \langle C^4(G) \rangle$

These imply that $a(G) = 4$. □

It may be remarked that the composition $G[K_s]$, (refer p. 108 of [2]) where $G = G(10)$ defined above and s is a positive integer, belongs to $\mathcal{G}$ and has vertex connectivity s and $a(G[K_s]) = 4$. Therefore vertex connectivity of G does not place severe restriction on $a(G)$ for $G \in \mathcal{G}$.

The following theorem, as pointed by the referee, is a corollary to a recent theorem of Chang-Nemhauser and a theorem of Laskar-Shier [9]. [Ref.: G. T. Chang and G. L. Nemhauser "K-domination and k-stability problems on graphs," SIAM Journal on Algebraic and Discrete Methods (to appear)]. This theorem was obtained earlier by Das-Rao [Ref.: P. Das and S. B. Rao, Abstracts AMS, Vol. 3 (1982), p. 285] by applying Lemma 1 to a vertex minimal counter example of a self-centered chordal graph of diameter 3, if one such exixts.

Theorem 12 : *There is no self-centered chordal graph G with* $(r, d) = (3, 3)$. *Consequently for a self-centred graph* G, $r(G) \leqslant 2$.

The Corollary 14 of the following Lemma 13 will be used in Theorem 15 which characterizes self-centered chordal graphs with radius 2. Lemma 13 was proved by Jamison (unpublished), as mentioned in [8], by using convexity in graphs. It can also be deduced from Dirac's theorem [5] (see also introduction).

Lemma 13 : *If* $G \in \mathcal{G}$ *and* $u \in V(G)$, *then there exists a* $u' \in S(G)$ *such that* $d_G\ (u, u') = e_G(u)$.

Corollary 14 : *If* $G \in \mathcal{G}$ *and* $(r, d) = (2, 2)$ *then* $|S(G)| \geqslant 3$.

We now present a characterization of self-centered chordal graphs G. If $r(G) \leqslant 1$, then G is a complete graph.

Theorem 15 : *If* $G \in \mathcal{G}$ *is of order* n, $\Delta(G) \leqslant n-2$, *then* G *is self-centered with* $r = 2$ *if and only if exactly one of the following holds.*

(a) *There exists a* $z \in S(G)$ *such that* $G-z$ *is a self-centered graph with radius* 2, $<N^1_G(z)>$ *is an absorbent clique of* $G-z$ *not containing a vertex of degree* $n-2$ *in* $G-z$.

(b) $|S(G)| \geqslant 3$ *and there exists a function* f *from* $S(G)$ *into* $D_{n-2}(G)$ *such that for every* $u \in S(G)$, $u \neq f(u)$ *and* $uf(u) \notin E(G)$, *where* $D_{n-2}(G)$ *is the set of vertices* v *of* G *with degree of* v *equal to* $n-2$.

Proof : Sufficiency : Let $G \in \mathcal{G}$, $\Delta(G) \leqslant n-2$. Then $r(G) \geqslant 2$. To show $(r, d) = (2, 2)$ it is enough to show that $d = 2$. If (a) holds, then $e_G(z) = 2 = e_G(x)$, for every $x \in V(G-z)$. Therefore $d = 2$. If (b) holds, then let $V_1 = f(S(G))$, $V_2 = S(G)$, $V_3 = V(G) - V_1 - V_2$. Then $<V_1>$ is complete, $<S>$ has no edges and $G[V_1, V_2] = K_{s,s} - I$ where $s = |S(G)|$ and I is a 1-factor of the complete bipartite graph, $K_{s,s}$. If now $u, v \in V(G)$, $u \neq v$, $uv \notin E(G)$, then $d_G(u, v) = 2$ unless both u, v are in V_2, in which case if $w \in S(G)$ and $w \neq u, v$, then $[u, f(w), v]$ is a path of length 2 in G and therefore again $d_G(u, v) = 2$. Thus $d(G) = 2$, completing the proof of the sufficiency.

To prove the necessity, let G be a self-centered chordal graph with $r = 2$. First note, since $\Delta(G) \leqslant n-2$ and G is connected that

$$S(G) \cap D_{n-2}(G) = \phi. \qquad \ldots \quad (15.1)$$

By Corollary 14, $|S(G)| \geqslant 3$ and also for any $z \in S(G)$, the clique $N^1_G(z)$ is absorbent in $G-z$, not containing a full degree vertex of $G-z$. ... (15.2)

Further, for any $z \in S(G)$, by hypothesis $d(G-z) \leqslant 2$.

If now (a) does not hold, then for any $z \in S(G)$, we have $r(G-z) = 1$ and therefore there exists a vertex $f(z) \in C(G-z)$ such that $zf(z) \notin E(G)$ and by (15.1) $d_G(f(z)) = n-2$, therefore (b) holds.

If (b) does not hold, then there exists a $z \in S(G)$ such that for every $x \notin N^1_G(z) \bigcup \{z\}$, we have $\deg_G(x) \leqslant n-3$ and therefore $e_{G-z}(x) \geqslant 2$. Since $G \in \mathcal{G}$ is a (2, 2) graph and $z \in S(G)$, it follows that $e_{G-z}(y) = 2$ for every $y \in V(G-z)$; and therefore $G-z$ is a (2, 2) graph.

If (a) and (b) both hold, then for a z satisfying (a), $G-z$ is a (2, 2) graph which then implies that degree of $f(z)$ is at most $n-3$, contradicting (b).

This completes the proof of the necessity. $\square$

Lemma 16 : *If H is a chordal graph with $(r(H), d(H)) = (2, 3)$ and $r(<C(H)>) \geqslant 2$, then $H = C(G)$ for some $G \in \mathcal{G}$.*

Proof : Let G be the graph obtained from H by attaching one pendant vertex at each of the vertices of $C(H)$. Then G is chordal $(r(G), d(G)) = (3, 4)$ and $C(G) = V(H)$. $\square$

Lemma 17 : *If $H \in \mathcal{G}$ and H has two vertex disjoint absorbent cliques C_1, C_2, then $H = C(G)$ for some $G \in \mathcal{G}$.*

Proof : Define a graph G as follows :

$$V(G) = V(H) \bigcup \{x_1, x_2, x_3, x_4\} \text{ where } x_i \notin V(H), \ 1 \leqslant i \leqslant 4.$$

$$E(G) = E(H) \bigcup \{x_1x/x \in C_1\} \bigcup \{x_2x/x \in C_2\} \bigcup \{x_1x_3, x_2x_4\}.$$

Since C_1, C_2 are absorbent cliques of $H \in \mathcal{G}$, it is easy to check that $G \in \mathcal{G}$ and

$$e_G(x) = \begin{cases} 3, & \text{if } x \in V(H), \\ 4, & \text{if } x = x_1, x_2, \\ 5, & \text{if } x = x_3, x_4. \end{cases}$$

Therefore $C(G) = V(H)$ and $<C(G)> = H$. $\square$

To prove Theorem 21 which characterizes chordal H with $(r, d) = (1, 2)$ such that $H = <C(G)>$, we need the following two Lemmas 18 and 19. In Lemma 18 we prove the existence of a representative set for a family of cliques covering the vertices of a chordal graph and satisfying a property similar to the odd cycle property in graphs (see p. 112, [1]).

Lemma 18 : *Let $\mathcal{F} = \{C_i : i \in I\}$ be a family of nonempty cliques covering the vertices of a chordal graph G satisfying :* (D1) *for every $i, j \in I$ with $C_i \bigcap C_j = \phi$ there are vertices $x \in C_i$, $y \in C_j$ such that $xy \in E(G)$.*

Then there exists a clique C of G with the property that for every $i \in I$,

$$C \cap C_i \neq \phi. \qquad \ldots \quad (18.1)$$

Proof: The proof is by induction on $p = |V(G)|$. If $p = 1$, then clearly the lemma holds. Assume that the lemma holds for all chordal graphs of order $p-1$. Let G be a chordal graph of order p and $\mathcal{F} = \{C_i : i \in I\}$ be a family of nonempty cliques covering $V(G)$ such that (D1) holds. We consider two cases.

Case (1). For some $j \in I$, $C_j = \{u\}$ and $u \in S(G)$.

In this case let $$C = N'_G(u) \cup \{u\}. \qquad \ldots \quad (18.2)$$

Then C is a clique. If $C \cap C_i = \phi$ for some $i \in I$, then $u \notin C_i$. By hypothesis of Case (1), $C_i \cap C_j = \phi$ and then by (D1) there exist vertices $x \in C_i$, $y \in C_j$ such that $xy \in E(G)$. Notice that $y = u$. Hence $x \in N'_G(u) \subseteq C$ by (18.2) and therefore $x \in C \cap C_i$, a contradiction. Thus (18.1) holds in this case for this C.

Case (2). *Negation of Case* (1). Let $w \in S(G)$ and $\mathcal{F}' = \{C'_i = C_i - \{w\} / i \in I\}$. Then by the hypothesis of case (2), $\mathcal{F}'$ is a family of nonempty cliques covering the vertices of the chordal graph $G-w$ of order $p-1$. We prove that the condition (D1) holds for $\mathcal{F}'$. Suppose that for some

$$i, j \in I,\ C'_i \cap C'_j = \phi. \qquad \ldots \quad (18.3)$$

If $C_i \cap C_j = \phi$, then by (D1) for $\mathcal{F}$ there exist elements $x \in C_i$, $y \in C_j$ such that $xy \in E(G)$. In this case if neither x nor y equals w, then $xy \in E(G-w)$. So we may assume that $x = w$. Choose a vertex $z \in C'_i$; then $z \neq w = x$ and since $w, z \in C_i$ a clique, $wz \in E(G)$. Since $wy \in E(G)$ and $w \in S(G)$, we have $zy \in E(G-w)$, where $z \in C'_i$ and $y \in C_j = C'_j$ as $C_i \cap C_j = \phi$ and $w \in C_i$.

If $C_i \cap C_j \neq \phi$, then by (18.3) and definition of $\mathcal{F}'$, we have

$$C_i \cap C_j = \{w\}. \qquad \ldots \quad (18.4)$$

Choose $x \in C'_i$, $y \in C'_j$. Since C_i, C_j are cliques by (18.4), $wx, wy \in E(G)$ and then by simpliciality of w in G, $xy \in E(G-w)$.

By inductive hypothesis, there is a clique C' of $G-w$ such that (18.1) is satisfied for $\mathcal{F}'$. Then $C = C'$ is a clique of G and by definition of $\mathcal{F}'$, C satisfies (18.1) for $\mathcal{F}$. $\square$

Using Lemma 18, we prove the following

Lemma 19 : *If $G \in \mathcal{G}$ and $(r, d) = (2, 3)$, then G has an absorbent clique C with $C \subseteq C(G)$.*

Proof: We consider two cases.

Case (1). $G_1 = <C(G)>$ *is not complete.* Let $x \in S(G_1)$. We show that the clique $C = N^1_{G_1}(x)$ is absorbent in G. Since G_1 is not complete and $x \in S(G_1)$, there exists a vertex $y \in C(G)$, $x \neq y$, such that $xy \notin E(G)$. Then $S = N^1_{G_1}(x) \in S(x, y, G_1)$ and by Corollary 3, $S \in S(x, y, G)$. Now for any $z \in V(G)$ with $z \notin S$, since $r(G) = 2$ and $x, y \in C(G)$, it follows that $d_G(x, z) \leqslant 2$ and $d_G(y, z) \leqslant 2$. Since $S \in S(x, y, G)$ it follows that z is adjacent in G to some vertex of S. Note that $S \subseteq C(G)$.

Case (2). $G_1 = <C(G)>$ *is complete.* We show that G_1 is absorbent in G by using Lemma 18 and the following observation for $G \in \mathcal{G}$: If D is a cycle of G, then for every edge uv of D there is a vertex $w \neq u, v$ in D such that

$$uw, wv \in E(G). \qquad \ldots \quad (19.1)$$

Suppose G_1 is not absorbent in G. Let $x_1, \ldots, x_p$ be a listing of the elements of $V(G)$; and

$$I = \{i/1 \leqslant i \leqslant p \text{ and } x_i \text{ is not absorbed by } G_1\}. \qquad \ldots \quad (19.2)$$

Since $r(G) = 2$, for every $i \in I$ the vertex x_i is at distance 2 from every vertex of $C(G)$. ... (19.3)

Claim 1: For $i \in I$, there exists a vertex $z_i \in V(G) - C(G)$ with $x_iz \in E(G)$ such that

$$zx \in E(G) \text{ for every } x \in C(G). \qquad \ldots \quad (19.4)$$

Choose a vertex $z \in V(G)$ with $x_iz \in E(G)$ such that z is adjacent to the maximum number of vertices in $C(G)$. Notice by (19.3) that $z \in V(G) - C(G)$ and z is adjacent to at least one vertex in $C(G)$. Suppose now that there exists an element $x \in C(G)$ such that $zx \notin E(G)$. By (19.3) there is a $z' \in V(G) - C(G)$ such that $z'x, z'x_i \in E(G)$. By the selection of z, then there exists a $x' \in C(G)$ with $z'x' \notin E(G)$ but $zx' \in E(G)$. Then $[z, x', x, z', x_i]$ is a 5-cycle of G such that $zx, x'z', x'x_i \notin E(G)$ contradicting (19.1) for the edge zx'.

For $i \in I$, define C_i to be the set of all vertices z in $V(G) - C(G)$ such that $x_iz \in E(G)$ and $zx \in E(G)$ for every $x \in C(G)$. ... (19.6)

We now prove that $\mathcal{F} = \{C_i : i \in I\}$ is a family of nonempty cliques of the chordal graph H, induced on $< \bigcup_{i \in I} C_i >$, satisfying condition (D1) of Lemma 18. That C_i is nonempty follows by claim 1. Let $z, z' \in C_i$, $z \neq z'$. If $zz' \notin E(G)$ for some $z, z' \in C_i$, $z \neq z'$, then for any $x \in C(G)$, $[z, x, z', x_i']$ is a

4-cycle by (19.4) of G without chords. Thus C_i is a clique of G and also the cliques of $\mathcal{F}$ together cover, by definition, $V(H)$.

Suppose now (D1) is not satisfied for C_i, C_j, $i, j \in I$. ... (19.6)

Then $$C_i \cap C_j = \phi. \quad \text{... (19.7)}$$

We consider cases depending on the value $d_G(x_i, x_j)$. Since $d(G) = 3$, $d_G(x_i, x_j) \leqslant 3$.

Case (a) $d_G(x_i, x_j) = 1$. Choose $z \in C_i$, $z' \in C_j$ and $x \in C(G)$. Then by (19.5), (19.6), (19.3) and the selection of x, $[x, z, x_i, x_j, z']$ is a 5-cycle of G in which $xx_i, xx_j, zz' \notin E(G)$, contradicting (19.1) for the edge xz.

Case (b) $d_G(x_i, x_j) = 2$. Let $[x_i, y, x_j]$ be an x_i—x_j geodesic in G. By (19.7) $y \notin C_i \cup C_j$ and since $x_iy \in E(G)$ but $y \notin C_i$ by (19.5), there exists an $x \in C(G)$ such that $yx \notin E(G)$. Choose $z \in C_i$ and $z' \in C_j$. Then by (19.3), (19.5), (19.6) and the selection of x, $[x, z, x_i, y, x_j, z']$ is a 6-cycle of G in which $xx_i, xy, xx_j, zz' \notin E(G)$, contradicting (19.1) for the edge xz.

Case (c) $d_G(x_i, x_j) = 3$. *Subcase* (1) : There exists an x_i—x_j geodesic $[x_i, y, y', x_j]$ with a vertex in $C_i \cup C_j$.

Notice by (19.7) that $y \notin C_j$, $y' \notin C_i$. Without loss of generality assume that $y \in C_i$, then since $x_jy' \in E(G)$ by (19.6), $y' \notin C_j$ and by (19.5) there exists an $x \in C(G)$ such that $y'x \notin E(G)$. Choose $z' \in C_j$. Then by (19.3), (19.5), (19.6) and the selection of x, $[x, y, y', x_j, z']$ is a 5-cycle of G in which xy', xx_j, $yz' \notin E(G)$, contradicting (19.1) for the edge xy.

Subcase (2) : There is no x_i—x_j geodesic as in subcase (1).

Let $[x_i, y, y', x_j]$ be an x_i—x_j geodesic of G. Choose $z \in C_i$, $z' \in C_j$. As $x_iy \in E(G)$ but $y \notin C_i$ by hypothesis, there is a vertex $x \in C(G)$ such that $yx \notin E(G)$. Then by (19.3), (19.5), (19.6) and the selection of x, and the hypothesis of subcase (2) $[x, z, x_i, y, y', x_j, z']$ is a 7-cycle of G in which xx_i, xy, xx_j, zz', $zy' \notin E(G)$, contradicting (19.1) for the edge xz.

Now we are ready to complete the proof. Thus $\mathcal{F}$ satisfies the hypothesis of the Lemma 18 for the chordal graph H induced on $<\bigcup_{i \in I} C_i>$. By Lemma 18 there exists a clique C of H such that (18.1) holds for every $i \in I$. Fix an $i \in I$ and choose a vertex $u \in C \cap C_i$. ... (19.8)

We show that $u \in C(G)$. Since $r(G) = 2$, for this it is enough to show that

for every $y \in V(G)$, $d_G(u, y) \leqslant 2$. ... (19.9)

If y is in $C(G)$ or is absorbed by $C(G)$, then by (19.8) and (19.5), the equation (19.9) holds. Thus we may assume that $y = x_j$ for some $j \in I$. By (18.1), there exists a vertex $v \in C \bigcap C_j$ and by (19.8), (19.5) and the fact C is a clique it follows that $[u, v, x_j = y]$ is a path of length 2 in G and (19.9) holds. Thus $u \in C(G)$ and by (19.8) and (19.5) we get a contradiction.

Therefore the assumption that the clique $C(G)$ is not absorbent is false. $\square$

Corollary 20 : *If $G \in \mathcal{G}$ is of order $\geqslant 2$ then $d(G) \leqslant 3$ if and only if G has an absorbent clique contained in $C(G)$.*

Proof : If G has an absorbent clique, then it is easy to see that $d(G) \leqslant 3$. The converse follows from Lemma 19 and the fact that if $d(G) \leqslant 2$, then for any $u \in S(G)$, $N_G^1(u)$ is an absorbent clique of G contained in $C(G)$. $\square$

In the next lemma we show that the converse of Lemma 17 holds when $H \in \mathcal{G}$, $r(H) = 1$ and $d(H) = 2$.

Theorem 21 : *If $H \in \mathcal{G}$ with $r(H) = 1$, $d(H) = 2$, then $H = \langle C(G) \rangle$ for some $G \in \mathcal{G}$ if and only if H has two vertex disjoint absorbent cliques; or equivalently $|C(H)| \geqslant 2$ or $|C(H)| = 1$ and $d(H - z) \leqslant 3$ where $C(H) = \{z\}$.*

Proof : The sufficiency follows from Lemma 17. The necessity may be proved as follows. If $|C(H)| \geqslant 2$, let x, y be distinct vertices of $C(H)$. Then $\{x\}$, $\{y\}$ are vertex disjoint absorbent cliques of H. Thus we may assume that $|C(H)| = 1$. Let $C(H) = \{z\}$. As $H = \langle C(G) \rangle$ it follows by Corollary 6 that z is not a cut vertex of H, as $d(H) = 2$ and $|V(H)| \geqslant 3$. Therefore $H_1 = H - z$ is connected.

Claim 1. $d(H_1) \leqslant 3$. Suppose $d(H_1) \geqslant 4$ and let $x, y \in V(H_1)$ such that $d_{H_1}(x, y) = d(H_1)$. Then

$$S_1 = N^2_{H_1}(x) \in S(x, y, H_1) \qquad \ldots \quad (21.1)$$

and therefore
$$S = S_1 \bigcup \{z\} \in S(x, y, H). \qquad \ldots \quad (21.2)$$

But $S(x, y, H) = S(x, y, \langle C(G) \rangle) \subseteq S(x, y, G)$ by Corollary 3. Therefore S contains an element $S_0 \in S_0(x, y, G)$. Then, by (21.1) and (21.2), for every $w \in S_0 - \{z\}$, we have

$$d_H(x, w) \geqslant 2 \quad \text{and} \quad d_H(y, w) \geqslant 2.$$

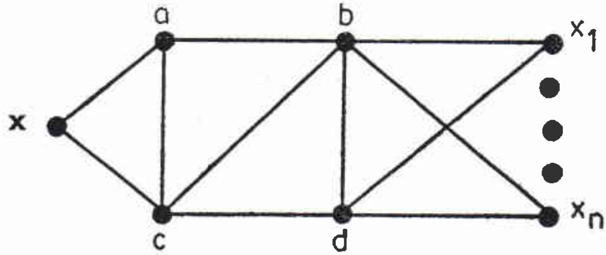
As $H = \langle C(G) \rangle$ and $S_0 - \{z\} \subseteq V(H) = C(G)$ by (21.1), it follows that for every $w \in S_0 - \{z\}$,

$$d_G(x, w) \geqslant 2 \quad \text{and} \quad d_G(y, w) \geqslant 2,$$

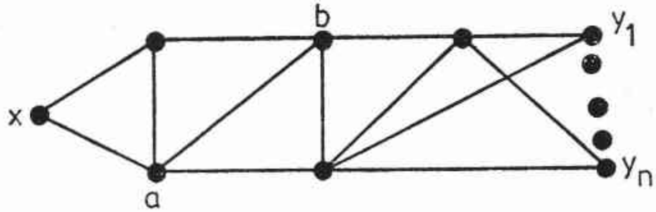
contradicting (C2) of Lemma 1 for S_0 and G. This completes the proof of the claim.

Now we are ready to complete the proof of the Theorem. By claim 1 and Corollary 20, $H_1 = H - \{z\}$ has an absorbent clique C_1, say. As $r(H) = 1$, $\{z\} = C(H)$, it follows that C_1 and $C_2 = \{z\}$ are vertex disjoint absorbent cliques of H. □

We give an infinite family $\mathscr{F}_1$ (respectively, $\mathscr{F}_2$) of $H \in \mathscr{G}$ with $(r(H), d(H)) = (2, 3)$ such that H is the center graph (respectively, not the center graph) of some (respectively, any) $G \in \mathscr{G}$.



The graph $\mathbf{H_1}$(n)



The graph $\mathbf{H_2}$(n)

$$\mathscr{F}_1 = \{H_1(n) : n \text{ is a positive integer}\}.$$

$$\mathscr{F}_2 = \{H_2(n) : n \text{ is a positive integer}\}.$$

Note that in $H_1(n)$, $C_1 = \{a, b\}$, $C_2 = \{c, d\}$ are two vertex disjoint absorbent cliques. Hence by Lemma 17, $\mathscr{F}_1$ has the stated property.

Note that in $H_2(n)$, $S_0 = \{a, b\} \in S_0(x, y_1, H_2(n))$. By (C2) of Lemma 1, $\mathscr{F}_2$ has the stated property.

We remark that Theorems 8, 12, 15, 21 and the fact that K_n is a self-centered chordal graph together characterize $H \in \mathscr{G}$ with $(r(H), d(H)) \neq (2, 3)$ which are the center graphs of some $G \in \mathscr{G}$. Further, Corollary 4 may be used to show that the center graph of a k-tree is either complete or is again a k-tree and then using Lemma 1, a good characterization of the center graphs of k-trees, in particular the center graphs of maximal outerplanar graphs [12] may be given. These are deferred to a later communication.

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